ISRAEL JOURNAL OF MATHEMATICS **150** (2005), 253–283

COFINALITY OF NORMAL IDEALS ON $P_\kappa(\lambda)$ II

BY

PIERRE MATET

Université de Caen-CNRS, Mathématiques, BP 5186
14032 Caen Cedex, France
e-mail: matet@math.unicaen.fr

AND

CÉDRIC PÉAN*

Université de Caen-CNRS, Mathématiques, BP 5186
14032 Caen Cedex, France

AND

SAHARON SHELAH**

Institute of Mathematics, The Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Department of Mathematics, Rutgers University
New Brunswick, NJ 08854, USA
e-mail: shelah@math.huji.ac.il

ABSTRACT

For an ideal J on an infinite set X with $\mathrm{add}(J) = \kappa$, let $\overline{\mathrm{cof}}(J)$ be the smallest size of any subfamily Y of J with the property that any member of J can be covered by less than κ members of Y. We study the value of $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A)$ for A in $(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$, where $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ denotes the smallest $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. We also discuss the problem of whether there exists a set A such that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda} \mid A$, or even $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} |A = I_{\kappa,\lambda} \mid A$.

* Some of the material in this paper originally appeared as part of the author's doctoral dissertation completed at the Université de Caen, 1998.

** Partially supported by the Israel Science Foundation. Publication 813.
Received July 10, 2000 and in revised form January 31, 2005

0. Introduction

In [7] we introduced the notion of a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. We gave necessary and sufficient conditions for the existence of such ideals and described the smallest one, denoted by $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. Furthermore, we determined the cofinality of this ideal. In the present paper the centre of our investigations is the reduced cofinality of $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. For an ideal J on $P_\kappa(\lambda)$, its reduced cofinality $\overline{\text{cof}}(J)$ is the smallest size of any subcollection Y of J such that every element of the ideal is covered by the union of less than κ many members of Y. This notion permits a finer analysis, as we have $\text{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A) = \text{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$ for all A, whereas there may exist a set A such that $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A) < \overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$. Moreover, it seems more appropriate than the classical notion of cofinality for handling situations when λ or δ is a singular cardinal of cofinality less than κ.

Johnson [4] was the first to show that there may exist a set A such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda} \mid A$. He was quickly followed by Baumgartner (see [4]) whose example is more widely applicable. Péan asked in his thesis whether it is consistent that $NS_{\kappa,\lambda} = I_{\kappa,\lambda} \mid A$ for some A. Donder, Koepke and Levinski [2] proved that there is no such A in case $cf(\lambda) \geq \kappa$, a fact which was rediscovered by Shelah [10] and Shioya [11]. Shelah [10] also obtained a positive result. Namely, he established that $NS_{\kappa,\lambda} = I_{\kappa,\lambda} \mid A$ for some A if λ is a strong limit cardinal of cofinality less than κ. So under GCH, there exists A such that $NS_{\kappa,\lambda} = I_{\kappa,\lambda} \mid A$ if and only if $cf(\lambda) < \kappa$. The present paper can be seen as a continuation of [10] in the more general framework of $[\delta]^{<\theta}$-normality. We use the concept of reduced cofinality as a tool for dealing with the question of whether there exists a set A such that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda} \mid A$, or even $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A = I_{\kappa,\lambda} \mid A$. We will give a complete answer to this question under GCH.

In Section 1 we review basic material concerning $[\delta]^{<\theta}$-normal ideals on $P_\kappa(\lambda)$. In Section 2 we list some simple properties of $\overline{\text{cof}}(J)$. Sections 3 and 4 are concerned with the evaluation of $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$ and $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A)$. We give an estimate for $\overline{\text{cof}}(NS_{\kappa,\lambda)}^{[\delta]^{<\theta}})$ in the case that $\kappa \leq \delta < \lambda$ and present some applications. We prove that $\overline{\text{cof}}(NS_{\kappa,\lambda}) = \lambda$ in the case that λ is a strong limit cardinal of cofinality less than κ. Furthermore, we show that if μ is a singular strong limit cardinal and κ is large enough, then $\overline{\text{cof}}(NS_{\kappa,\lambda}^{\mu}) = \lambda$. We also establish some lower bounds. In particular we prove that if $A \in NS_{\kappa,\lambda}^{*}$, then $\overline{\text{cof}}(NS_{\kappa,\lambda}^{\kappa} \mid A) \geq \lambda$. Moreover, this inequality is strict in case $cf(\lambda) = \kappa$.

Sections 5 and 6 deal with the problem of whether there exists a set A such that $\overline{\text{cof}}(I_{\kappa,\lambda} \mid A) < \overline{\text{cof}}(I_{\kappa,\lambda})$. We show that if $\lambda \leq \kappa^{+\omega}$ or $A \in NS_{\kappa,\lambda}^{*}$, then $\overline{\text{cof}}(I_{\kappa,\lambda} \mid A) = \lambda$. For $\kappa = \omega_1$, $\lambda = \kappa^{+(\omega+1)}$ and $\sigma = \kappa^{+\omega}$, we establish the following: (a) If either $\square_\sigma^*$ holds, or $2^{<\kappa} < \sigma$ and $\lambda < \sigma^{<\kappa}$, then there is $A \in NS^+_{\kappa,\lambda}$ such that $\overline{\text{cof}}(I_{\kappa,\lambda} \mid A) = \sigma$. (b) $\lambda \longrightarrow [\kappa]^2_{\sigma,<\kappa}$ implies that $\overline{\text{cof}}(I_{\kappa,\lambda}|A) = \lambda$ for all $A \in I^+_{\kappa,\lambda}$. In Section 7 we give a necessary and sufficient condition for the existence of a set A such that $NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = I_{\kappa,\lambda} \mid A$. We sum up the situation under GCH in a table that lists all quadruples $(\kappa, \lambda, \delta, \theta)$ such that $NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \mid A = I_{\kappa,\lambda} \mid A$ for some A. We also discuss the problem of whether, for $\delta \geq \kappa$, there exists a set A such that $\overline{\text{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \mid A) < \lambda$. Finally, in Section 8, we show that if GCH holds, $\delta \geq \kappa$ and P is the notion of forcing for adding $(\lambda^{<\kappa})^+$ Cohen subsets of κ, then in V^P, we have $NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \mid A \neq I_{\kappa,\lambda} \mid A$ for all A.

1. $[\delta]^{<\theta}$-normal ideals on $P_\kappa(\lambda)$

In this section we review basic definitions concerning $[\delta]^{<\theta}$-normal ideals on $P_\kappa(\lambda)$, as well as various results which will be used in later sections, often without quoting them.

Given a set A and a cardinal τ, we let $P_\tau(A) = [A]^{<\tau} = \{a \subseteq A : |a| < \tau\}$.

Throughout the paper κ denotes a regular uncountable cardinal, and λ a cardinal with $\lambda \geq \kappa$.

For $a \in P_\kappa(\lambda)$, we set $\widehat{a} = \{b \in P_\kappa(\lambda) : a \subseteq b\}$.

$I_{\kappa,\lambda}$ is the set of all $B \subseteq P_\kappa(\lambda)$ such that $B \cap \widehat{a} = \emptyset$ for some $a \in P_\kappa(\lambda)$.

By an ideal on $P_\kappa(\lambda)$ we mean a collection J of subsets of $P_\kappa(\lambda)$ such that (i) $P(B) \subseteq J$ for all $B \in J$, (ii) $\cup Y \in J$ for all $Y \subseteq J$ with $0 < |Y| < \kappa$, (iii) $I_{\kappa,\lambda} \subseteq J$, and (iv) $P_\kappa(\lambda) \notin J$.

Given an ideal J on $P_\kappa(\lambda)$, we let

$$J^+ = P(P_\kappa(\lambda)) - J \quad \text{and} \quad J^* = \{B \subseteq P_\kappa(\lambda) : P_\kappa(\lambda) - B \in J\}.$$

For $A \in J^+$, we let $J \mid A = \{B \subseteq P_\kappa(\lambda) : B \cap A \in J\}$. cof$(J)$ is the least cardinality of any $S \subseteq J$ with $J = \bigcup_{B \in S} P(B)$.

It is simple to see that $I_{\kappa,\lambda}$ is an ideal on $P_\kappa(\lambda)$. $u(\kappa, \lambda)$ is the least cardinality of any $A \subseteq P_\kappa(\lambda)$ with $A \in I^+_{\kappa,\lambda}$.

Proposition 1.1 ([7]):

(i) $u(\kappa, \lambda) \geq \lambda$.

(ii) $\text{cof}(I_{\kappa,\lambda} \mid A) = u(\kappa, \lambda)$ *for every* $A \in I^+_{\kappa,\lambda}$.

Given four cardinals τ, ρ, χ and σ, let $\mathfrak{X}(\tau, \rho, \chi, \sigma)$ be the set of all $X \subseteq P_\rho(\tau)$ with the property that for every $a \in P_\chi(\tau)$, there is $x \in P_\sigma(X) \setminus \{\emptyset\}$ such that

$a \subseteq \cup x$. If $\mathfrak{X}(\tau,\rho,\chi,\sigma) \neq \emptyset$, then we let $\text{cov}(\tau,\rho,\chi,\sigma)$ be the least cardinality of any member X of $\mathfrak{X}(\tau,\rho,\chi,\sigma)$.

PROPOSITION 1.2 ([8]):

(i) *Let τ be a cardinal. Then* $\text{cov}(\tau,\tau^+,\tau^+,2) = 1$.

(ii) *Let τ be a regular infinite cardinal, and σ be a cardinal with $2 \leq \sigma \leq \tau$. Then* $\text{cov}(\tau,\tau,\tau,\sigma) = \tau$.

(iii) *Let τ be an infinite cardinal, ρ be a cardinal with $2 \leq \rho < \tau$, and σ be a cardinal with $\sigma \geq 2$. Then* $\text{cov}(\tau,\rho,2,\sigma) \geq \tau$.

PROPOSITION 1.3 ([8]):

(i) $\text{cov}(\lambda,\kappa,\kappa,2) = u(\kappa,\lambda)$.

(ii) *Let ρ be a cardinal with $\kappa \leq \rho \leq \lambda$. Then*

$$\text{cov}(\lambda^+,\rho,\rho,\kappa) = \lambda^+ \cdot \text{cov}(\lambda,\rho,\rho,\kappa).$$

(iii) *Let ρ be a cardinal with $\kappa \leq \rho < \lambda$. Assume that λ is a limit cardinal and either $cf(\lambda) < \kappa$, or $cf(\lambda) \geq \rho$. Then*

$$\text{cov}(\lambda,\rho,\rho,\kappa) = \sup_{\rho<\tau<\lambda} \text{cov}(\tau,\rho,\rho,\kappa).$$

(iv) *Let ρ be a cardinal such that $cf(\rho) < \kappa < \rho < \lambda$. Then*

$$\text{cov}(\lambda,\rho,\rho,\kappa) = \text{cov}(\lambda,\rho,\rho^+,\kappa).$$

COROLLARY 1.4: *Let ρ be a regular cardinal such that $\kappa < \rho \leq \lambda < \rho^{+\kappa}$. Then* $\text{cov}(\lambda,\rho,\rho,\kappa) = \lambda$.

Throughout the paper δ denotes an ordinal with $1 \leq \delta \leq \lambda$, and θ a cardinal with $2 \leq \theta \leq \kappa$.

We set $\overline{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and κ is a limit cardinal, and $\overline{\theta} = \nu$ if $\theta = \kappa = \nu^+$.

Given $X_e \subseteq P_\kappa(\lambda)$ for $e \in P_\theta(\delta)$, we let

$$\nabla_{e\in P_\theta(\delta)} X_e = \bigcup_{e\in P_\theta(\delta)} \{a \in X_e : e \in P_{|a\cap\theta|}(a \cap \delta)\}.$$

Given an ideal J on $P_\kappa(\lambda)$, $\nabla^{[\delta]^{<\theta}} J$ is the set of all $B \subseteq P_\kappa(\lambda)$ for which one can find $B_e \in J$ for $e \in P_\theta(\delta)$ so that

$$B \subseteq \{a \in P_\kappa(\lambda) : a \cap \theta = \emptyset\} \cup (\nabla_{e\in P_\theta(\delta)} B_e).$$

We say that J is $[\delta]^{<\theta}$-normal if $J = \nabla^{[\delta]^{<\theta}} J$.

PROPOSITION 1.5 ([7]):

(i) *Assume that $\delta < \kappa$, or $\theta < \kappa$, or κ is not a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if $|P_{\overline{\theta}}(\mu)| < \kappa$ for every cardinal $\mu < \kappa \cap (\delta + 1)$.*

(ii) *Assume that $\delta \geq \kappa$, $\theta = \kappa$ and κ is a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if κ is Mahlo.*

(iii) *Assume that $\delta \geq \kappa$ and there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. Then $\kappa^{<\overline{\theta}} = \kappa$. Moreover, $(\mu^{<\overline{\theta}})^{<\overline{\theta}} = \mu^{<\overline{\theta}}$ for every cardinal $\mu > \kappa$.*

If there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, then $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ denotes the smallest such ideal.

PROPOSITION 1.6 ([7]):

(i) $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = \nabla^{[\delta]^{<\overline{\theta}\cdot 3}} I_{\kappa,\lambda}$.

(ii) $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{[\delta]^{<\theta\cdot\aleph_0}}$.

(iii) *If $\delta < \kappa$, then* $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}$.

For $f\colon P_\theta(\delta) \longrightarrow P_\kappa(\lambda)$, $C_f^{\kappa,\lambda}$ denotes the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \theta \neq \emptyset$ and $f(e) \subseteq a$ for every $e \in P_{|a\cap\theta|}(a \cap \delta)$.

PROPOSITION 1.7 ([7]):

(i) *Given $B \subseteq P_\kappa(\lambda)$, $B \in NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ if and only if $B \cap C_f^{\kappa,\lambda} = \emptyset$ for some $f\colon P_{\theta\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$.*

(ii) *Suppose $\delta \geq \kappa$. Then given $B \subseteq P_\kappa(\lambda)$, $B \in NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ if and only if $B \cap \{a \in C_g^{\kappa,\lambda} : a \cap \kappa \in \kappa\} = \emptyset$ for some $g\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_3(\lambda)$.*

Given $X_\alpha \subseteq P_\kappa(\lambda)$ for $\alpha < \delta$, we let $\nabla_{\alpha<\delta} X_\alpha = \bigcup_{\alpha<\delta}(X_\alpha \cap \widehat{\{\alpha\}})$.

Given an ideal J on $P_\kappa(\lambda)$, $\nabla^\delta J$ denotes the set of all $B \subseteq P_\kappa(\lambda)$ for which one can find $B_\alpha \in J$ for $\alpha < \delta$ so that $B \subseteq (P_\kappa(\lambda) \setminus \widehat{\{0\}}) \cup \nabla_{\alpha<\delta} B_\alpha$. J is called δ-normal if $J = \nabla^\delta J$.

$NS_{\kappa,\lambda}^\delta$ denotes the smallest δ-normal ideal on $P_\kappa(\lambda)$.

Note that $NS_{\kappa,\lambda}^\lambda = NS_{\kappa,\lambda}$.

PROPOSITION 1.8 ([7]): $NS_{\kappa,\lambda}^\delta = NS_{\kappa,\lambda}^{[\delta]^{<2}}$.

Given a cardinal $\mu > 0$, $\mathfrak{d}_{\kappa,\lambda}^\mu$ is the least cardinality of any family $\mathcal{F}$ of functions from μ to $P_\kappa(\lambda)$ such that for every $g\colon \mu \longrightarrow P_\kappa(\lambda)$, there is $f \in \mathcal{F}$ with the property that $g(\alpha) \subseteq f(\alpha)$ for every $\alpha < \mu$.

PROPOSITION 1.9 ([7]): *Let $\mu > 0$ be a cardinal. Then*

(i) $\mathfrak{d}^{\mu}_{\kappa,\lambda} \geq u(\kappa,\lambda)$.

(ii) $cf(\mathfrak{d}^{\mu}_{\kappa,\lambda}) > \mu$.

(iii) *If* $\mu \geq \kappa$ *and* $\lambda \geq 2^{\mu}$, *then* $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \lambda^{\mu}$.

(iv) *If* $\lambda \geq 2^{<\kappa}$, *then* $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \mathfrak{d}^{\mu}_{\kappa,\lambda^{<\kappa}}$.

PROPOSITION 1.10 ([7]):

(i) *If J is a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, then* $\mathrm{cof}(J) \geq \mathfrak{d}^{|P_{\bar{\theta}}(\delta)|}_{\kappa,\lambda}$.

(ii) $\mathrm{cof}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \mid A) = \mathfrak{d}^{|P_{\bar{\theta}}(\delta)|}_{\kappa,\lambda}$ *for every* $A \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^{+}$.

2. $\overline{\mathrm{cof}}(J)$

In this section we introduce the notion of the reduced cofinality $\overline{\mathrm{cof}}(J)$ of an ideal J on $P_\kappa(\lambda)$.

Definition: Given an ideal J on $P_\kappa(\lambda)$, $\overline{\mathrm{cof}}(J)$ is the least cardinality of any $Z \subseteq J$ such that for every $B \in J$, there is $x \in P_\kappa(Z)$ with $B \subseteq \bigcup x$.

The following collects some elementary facts.

PROPOSITION 2.1: *Let J be an ideal on $P_\kappa(\lambda)$. Then*

(i) $\kappa \leq \overline{\mathrm{cof}}(J) \leq \mathrm{cof}(J) \leq u(\kappa, \overline{\mathrm{cof}}(J))$.

(ii) *If* $\overline{\mathrm{cof}}(J) \leq \lambda$, *then* $\mathrm{cof}(J) = u(\kappa,\lambda)$.

(iii) $\overline{\mathrm{cof}}(J) \leq \lambda^{<\kappa}$ *if and only if* $\mathrm{cof}(J) \leq \lambda^{<\kappa}$.

(iv) $\overline{\mathrm{cof}}(J \mid A) \leq \overline{\mathrm{cof}}(J)$ *for all* $A \in J^{+}$.

Proof: The proofs of (i) and (iv) are easy and left to the reader. It is simple to see that $\mathrm{cof}(J) \geq u(\kappa,\lambda)$. Part (ii) follows from this and (i). For (iii), use (i) and the fact that $u(\kappa,\lambda^{<\kappa}) = \lambda^{<\kappa}$. ∎

PROPOSITION 2.2: *Assume $\lambda \leq \kappa^{+\omega}$. Then $\overline{\mathrm{cof}}(J) \geq \lambda$ for any ideal J on $P_\kappa(\lambda)$.*

Proof: Set $\lambda = \kappa^{+\alpha}$, and let J be an ideal on $P_\kappa(\lambda)$. Proposition 2.1 gives $\overline{\mathrm{cof}}(J) \geq \kappa$, so the result is immediate in case $\alpha = 0$. Now suppose $\alpha > 0$. We have $\mathrm{cof}(J) \geq u(\kappa,\kappa^{+\alpha}) \geq \kappa^{+\alpha}$. It is well-known (see e.g. Corollary 4.2 in [3]) that $u(\kappa,\kappa^{+n}) = \kappa^{+n}$ for all $n \in \omega$. We can conclude, using Proposition 2.1, that $\overline{\mathrm{cof}}(J) > \kappa^{+n}$ for every $n \in \alpha$.

3. $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$

It was shown in [7] that for $\delta \geq \kappa$,

$$\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \mathrm{cof}(NS_{\kappa,|\delta|}^{[|\delta|]^{<\theta}}) \cdot \mathrm{cov}\,(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, 2).$$

Now we establish a similar formula for $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$.

Proposition 3.1: *Assume $\delta \geq \kappa$. Then*

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \overline{\mathrm{cof}}(NS_{\kappa,|\delta|}^{[|\delta|]^{<\theta}}) \cdot \mathrm{cov}\,(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, \kappa).$$

Proof: Since $NS_{\kappa,\lambda}^{[\kappa+\xi]^{<\theta}} = NS_{\kappa,\lambda}^{[\kappa]^{<\theta}}$ for every $\xi < \kappa$ ([7]), we can assume w.l.o.g. that $\delta = \kappa$ or $\delta \geq \kappa + \kappa$. Select a bijection $j\colon |\delta| \longrightarrow \delta$ so that $j(\alpha) = \alpha$ for all $\alpha < \kappa$ in case $\delta = \kappa$ or $\delta \geq \kappa^+$, and let i denote its inverse. Define $v\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\delta)$ so that

(i) If $\overline{\theta} < \kappa$, then $v(e) = (\overline{\theta}\cdot 3) \cup j[\overline{\theta}\cdot 3]$.
(ii) If $\kappa = \overline{\theta}$ and either $\delta = \kappa$ or $\delta \geq \kappa^+$, then $v(e) = \{0\}$.
(iii) Suppose $\kappa = \overline{\theta}$ and $\kappa < \delta < \kappa^+$. Pick a bijection $q\colon \kappa \longrightarrow \delta \setminus \kappa$. Now for $\beta \in \kappa$, let $v(\{\beta\}) = \omega \cup \{q(\beta)\}$ and $v(\{q(\beta)\}) = \omega \cup \{\beta\}$.

For $a \in P_\kappa(\lambda)$, set $\overline{a} = i[a \cap \delta]$. Now let $a \in C_v^{\kappa,\lambda}$. If $\overline{\theta} < \kappa$, then $\overline{\theta}\cdot 3 \subseteq \overline{a}$ and $\overline{\theta}\cdot 3 \subseteq a$. If $\theta = \kappa$ and either $\delta = \kappa$ or $\delta \geq \kappa^+$, then $\overline{a} \cap (\overline{\theta}\cdot 3) = a \cap (\overline{\theta}\cdot 3)$. If $\overline{\theta} = \kappa$ and $\kappa < \delta < \kappa^+$, then we have $\omega \subseteq a$ and $q[a \cap \kappa] = (a \cap \delta) \setminus \kappa$, consequently $|\overline{a} \cap (\overline{\theta}\cdot 3)| = |\overline{a}| = |a \cap \delta| = |a \cap (\overline{\theta}\cdot 3)|$. So in any case we get $|\overline{a} \cap (\overline{\theta}\cdot 3)| = |a \cap (\overline{\theta}\cdot 3)|$.

Claim 1: $\overline{\mathrm{cof}}(NS_{\kappa,|\delta|}^{[|\delta|]^{<\theta}}) \leq \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$.

For the proof of Claim 1, select a family $\mathfrak{X}$ of functions from $P_{\overline{\theta}\cdot 3}(\delta)$ to $P_\kappa(\lambda)$ so that $|\mathfrak{X}| = \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$ and for every $h\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$, there is $X \in P_\kappa(\mathfrak{X}) \setminus \{\emptyset\}$ with $\bigcap_{g\in X} C_g^{\kappa,\lambda} \subseteq C_h^{\kappa,\lambda}$. For $f\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$, define $\widetilde{f}\colon P_{\overline{\theta}\cdot 3}(|\delta|) \longrightarrow P_\kappa(|\delta|)$ by $\widetilde{f}(u) = i[\delta \cap f(j[u])]$. Now fix $h\colon P_{\overline{\theta}\cdot 3}(|\delta|) \longrightarrow P_\kappa(|\delta|)$.

Set $A = \{a \in P_\kappa(\lambda) : \overline{a} \in C_h^{\kappa,|\delta|}\}$. Define $\overline{h}\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\delta)$ by $\overline{h}(e) = j[h(i[e])]$. Given $a \in C_v^{\kappa,\lambda} \cap C_{\overline{h}}^{\kappa,\lambda}$ and $u \in P_{|\overline{a}\cap(\overline{\theta}\cdot 3)|}(\overline{a})$, we have $\overline{h}(j[u]) \subseteq a \cap \delta$ since $j[u] \in P_{|a\cap(\overline{\theta}\cdot 3)|}(a \cap \delta)$, and therefore $h(u) \subseteq \overline{a}$. Hence $C_v^{\kappa,\lambda} \cap C_{\overline{h}}^{\kappa,\lambda} \subseteq A$. Thus $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^*$, so there is $X \in P_\kappa(\mathfrak{X}) \setminus \{\emptyset\}$ with $\bigcap_{g\in X} C_g^{\kappa,\lambda} \subseteq A$.

Let $d \in C_{\widetilde{v}}^{\kappa,|\delta|} \cap (\bigcap_{g\in X} C_{\widetilde{g}}^{\kappa,|\delta|})$. If $\overline{\theta} < \kappa$, then $\overline{\theta}\cdot 3 \subseteq d$ and $\overline{\theta}\cdot 3 \subseteq j[d]$. If $\overline{\theta} = \kappa$ and either $\delta = \kappa$ or $\delta \geq \kappa^+$, then $d \cap (\overline{\theta}\cdot 3) = j[d] \cap (\overline{\theta}\cdot 3)$. If $\overline{\theta} = \kappa$ and $\kappa < \delta < \kappa^+$, then $\omega \subseteq j[d]$ and $q[j[d] \cap \kappa] = j[d] \setminus \kappa$, hence

$|d \cap (\overline{\theta}\cdot 3)| = |d| = |j[d]| = |j[d] \cap (\overline{\theta}\cdot 3)|$. In any case $|d \cap (\overline{\theta}\cdot 3)| = |j[d] \cap (\overline{\theta}\cdot 3)|$. Now set

$$a = j[d] \cup \left(\bigcup \{g(e) : g \in X \text{ and } e \in P_{|j[d] \cap (\overline{\theta}\cdot 3)|}(j[d])\} \right).$$

Then $|a| < \kappa$ by Proposition 1.5. It is simple to see that $\overline{a} = d$ and $a \in \bigcap_{g \in X} C_g^{\kappa,\lambda}$. Hence $d \in C_h^{\kappa,|\delta|}$. Thus

$$C_{\widetilde{v}}^{\kappa,|\delta|} \cap \left(\bigcap_{g \in X} C_{\widetilde{g}}^{\kappa,|\delta|} \right) \subseteq C_h^{\kappa,|\delta|}.$$

Now we can conclude that $\overline{\mathrm{cof}}(NS_{\kappa,|\delta|}^{[|\delta|]^{<\theta}}) \leq |\mathfrak{X}|$. This completes the proof of Claim 1.

Claim 2: $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \leq \overline{\mathrm{cof}}(NS_{\kappa,|\delta|}^{[|\delta|]^{<\theta}}) \cdot \mathrm{cov}\,(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, \kappa)$.

For the proof of Claim 2, set $\sigma = \lambda \cap |\delta|^{<\overline{\theta}}$. Select $\mathcal{Z} \subseteq \{z \subseteq \lambda : |z| = \sigma\}$ so that $|\mathcal{Z}| = \mathrm{cov}(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, \kappa)$ and for every $y \in P_{\sigma^+}(\lambda)$, there is $Z \in P_\kappa(\mathcal{Z})$ with $y \subseteq \cup Z$. For $z \in Z$, pick a one-to-one t_z: $z \longrightarrow P_{\overline{\theta}\cdot 3}(\delta)$ and define k_z: $P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$ so that $k_z(t_z(\beta)) = \{\beta\}$ for all $\beta \in z$.

Select a family $\mathcal{H}$ of functions from $P_{\overline{\theta}\cdot 3}(|\delta|)$ to $P_\kappa(|\delta|)$ so that $|\mathcal{H}| = \overline{\mathrm{cof}}(NS_{\kappa,|\delta|}^{[|\delta|]^{<\theta}})$ and for every t: $P_{\overline{\theta}\cdot 3}(|\delta|) \longrightarrow P_\kappa(|\delta|)$, there is $H \in P_\kappa(\mathcal{H}) \setminus \{\emptyset\}$ with $\bigcap_{h \in H} C_h^{\kappa,|\delta|} \subseteq C_t^{\kappa,|\delta|}$. For $h \in H$, define $\widetilde{h}$: $P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\delta)$ by $\widetilde{h}(e) = j[h(i[e])]$.

Now fix g: $P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$. Pick $Z \in P_\kappa(\mathcal{Z})$ so that $\cup \mathrm{rang}(g) \subseteq \cup Z$. Put $r_e = \bigcup_{z \in Z} \bigcup_{\beta \in z \cap g(e)} t_z(\beta)$ for $e \in P_{\overline{\theta}\cdot 3}(\delta)$. Define g': $P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\delta)$ by: $g'(e) = r_e$ if $\overline{\theta} < \kappa$, and $g'(e) = r_e \cup |r_e|^+$ otherwise. Also, define g'': $P_{\overline{\theta}\cdot 3}(|\delta|) \longrightarrow P_\kappa(|\delta|)$ by $g''(x) = i[g'(j[x])]$. Select $H \in P_\kappa(\mathcal{H}) \setminus \{\emptyset\}$ so that $\bigcap_{h \in H} C_h^{\kappa,|\delta|} \subseteq C_{g''}^{\kappa,|\delta|}$. Now let $a \in C_v^{\kappa,\lambda} \cap (\bigcap_{h \in H} C_{\widetilde{h}}^{\kappa,\lambda}) \cap (\bigcap_{z \in Z} C_{k_z}^{\kappa,\lambda})$ and $e \in P_{|a \cap (\overline{\theta}\cdot 3)|}(a \cap \delta)$. Clearly $\overline{a} \in \bigcap_{h \in H} C_h^{\kappa,|\delta|}$, hence $\overline{a} \in C_{g''}^{\kappa,|\delta|}$. From this we can infer that $g'(e) \subseteq a \cap \delta$. So given $z \in Z$, we get that for each $\beta \in z \cap g(e)$, $t_z(\beta) \in P_{|a \cap (\overline{\theta}\cdot 3)|}(a \cap \delta)$, hence $k_z(t_z(\beta)) \subseteq a$. Thus $z \cap g(e) \subseteq a$. Since $g(e) = \bigcup_{z \in Z}(z \cap g(e))$, this gives $g(e) \subseteq a$. Thus

$$C_v^{\kappa,\lambda} \cap \left(\bigcap_{h \in H} C_{\widetilde{h}}^{\kappa,\lambda} \right) \cap \left(\bigcap_{z \in Z} C_{k_z}^{\kappa,\lambda} \right) \subseteq C_g^{\kappa,\lambda}.$$

It easily follows that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \leq |\mathcal{H}| \cdot |\mathcal{Z}|$. This completes the proof of Claim 2.

CLAIM 3: $\mathrm{cov}(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, \kappa) \le \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$.

To prove Claim 3, select a family $\mathfrak{X}$ of functions from $P_{\overline{\theta}\cdot 3}(\delta)$ to $P_\kappa(\lambda)$ so that $|\mathfrak{X}| = \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$ and for every $h\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$, there exists $X \in P_\kappa(\mathfrak{X}) \setminus \{\emptyset\}$ with $\bigcap_{g\in X} C_g^{\kappa,\lambda} \subseteq C_h^{\kappa,\lambda}$. For $g \in X$, set $B_g = \delta \cup (\cup \mathrm{ran}(g))$. Notice that $|B_g| \le |\delta|^{<\overline{\theta}}$.

Now fix $A \subseteq \lambda$ with $|A| \le |\delta|^{<\overline{\theta}}$. Pick $h\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$ so that $A \subseteq \cup\mathrm{ran}(h)$. Then there is $X \in P_\kappa(\mathfrak{X}) \setminus \{\emptyset\}$ such that $\bigcap_{g\in X} C_g^{\kappa,\lambda} \subseteq C_h^{\kappa,\lambda}$. For $e \in P_{\overline{\theta}\cdot 3}(\delta)$, define $z_e \in P_\kappa(\lambda)$ as follows. First suppose $\overline{\theta} < \kappa$. Put $\rho = \overline{\theta}\cdot\aleph_0$ if $\overline{\theta}\cdot\aleph_0$ is a regular cardinal, and $\rho = (\overline{\theta}\cdot\aleph_0)^+$ otherwise. Define s_α for $\alpha < \rho$ by: $s_0 = e \cup \rho$ and for $\alpha > 0$,

$$s_\alpha = \bigcup_{\beta<\alpha} s_\beta \cup \bigcup\Big\{g(d) : g \in X \text{ and } d \in P_{\overline{\theta}\cdot 3}\Big(\Big(\bigcup_{\beta<\alpha} s_\beta\Big) \cap \delta\Big)\Big\}.$$

Now let $z_e = \bigcup_{\alpha<\rho} s_\alpha$. Next suppose $\overline{\theta} = \kappa$. Define y_α and ξ_α for $\alpha < \kappa$ by:

(0) $y_0 = e \cup |e|^+ \cup \omega$.

(1) $\xi_\alpha = \cup(y_\alpha \cap \kappa)$.

(2) $y_{\alpha+1} = y_\alpha \cup (\xi_\alpha + 2) \cup \bigcup\{g(d) : g \in X \text{ and } d \subseteq y_\alpha \cap \delta\}$.

(3) $y_\alpha = \bigcup_{\beta<\alpha} y_\beta$ if α is an infinite limit ordinal.

Select a regular infinite cardinal τ so that $\xi_\tau = \tau$. Now let $z_e = y_\tau$. It is simple to see that $z_e \in \bigcap_{g\in X} C_g^{\kappa,\lambda}$ and $e \in P_{|z_e \cap (\overline{\theta}.3)|}(z_e \cap \delta)$. Hence $z_e \in C_h^{\kappa,\lambda}$ and $h(e) \subseteq z_e \subseteq \bigcup_{g\in X} B_g$. So $A \subseteq \bigcup_{e\in P_{\overline{\theta}\cdot 3}(\delta)} h(e) \subseteq \bigcup_{g\in X} B_g$. It follows that $\mathrm{cov}(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, \kappa) \le |\mathfrak{X}|$. This completes the proof of Claim 3. ∎

COROLLARY 3.2: *Assume $\delta \ge \kappa$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[|\delta|]^{<\theta}})$.*

COROLLARY 3.3: *Let μ be a cardinal with $\kappa \le \mu < \lambda$. Then*

(i) *If $\mu^{<\overline{\theta}} \ge \lambda$, then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}) = \overline{\mathrm{cof}}(NS_{\kappa,\mu}^{[\mu]^{<\theta}})$.*

(ii) *If $\mu^{<\overline{\theta}} < \lambda < (\mu^{<\overline{\theta}})^{+\kappa}$, then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}) = \lambda \cdot \overline{\mathrm{cof}}(NS_{\kappa,\mu}^{[\mu]^{<\theta}})$.*

(iii) *If $\mu^{<\overline{\theta}} \le \lambda$, then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda^+}^{[\mu]^{<\theta}}) = \lambda^+ \cdot \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}})$.*

(iv) *If λ is a limit cardinal and either $cf(\lambda) < \kappa$ or $cf(\lambda) > \mu^{<\overline{\theta}}$, then*

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}) = \sup_{\mu<\tau<\lambda} \overline{\mathrm{cof}}(NS_{\kappa,\tau}^{[\mu]^{<\theta}}).$$

Proof: Use Propositions 1.2 and 1.3 and Corollary 1.4. ∎

It follows from (i) that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda^{<\overline{\theta}}}^{[\lambda]^{<\theta}}) = \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})$. Concerning (iv), let us remark that (by Corollary 4.6 below) if μ is a cardinal with $\kappa \le \mu < \lambda$, and λ a strong limit cardinal with $\kappa \leq cf(\lambda) \leq \mu^{<\overline{\theta}}$, then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}) > \sup_{\mu<\tau<\lambda} \overline{\mathrm{cof}}(NS_{\kappa,\tau}^{[\mu]^{<\theta}})$.

COROLLARY 3.4: *Let μ and ρ be two cardinals with $\kappa \leq \mu \leq \rho < \lambda$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}) \geq \overline{\mathrm{cof}}(NS_{\kappa,\rho}^{[\mu]^{<\theta}})$.*

It can also be shown that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda})$ increases with λ.

PROPOSITION 3.5: *Let ρ be a cardinal with $\kappa \leq \rho < \lambda$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) \geq \overline{\mathrm{cof}}(NS_{\kappa,\rho})$.*

Proof: Select a family $\mathcal{K}$ of functions from $P_\omega(\lambda)$ to $P_\kappa(\lambda)$ so that $|\mathcal{K}| = \overline{\mathrm{cof}}(NS_{\kappa,\lambda})$ and for every h: $P_\omega(\lambda) \longrightarrow P_\kappa(\lambda)$, there is $K \in P_\kappa(\mathcal{K}) \setminus \{\emptyset\}$ with $\bigcap_{k\in K} C_k^{\kappa,\lambda} \subseteq C_h^{\kappa,\lambda}$. For $u \in P_\omega(\mathcal{K}) \setminus \{\emptyset\}$, define u^*: $P_\omega(\lambda) \longrightarrow P_\kappa(\lambda)$ by $u^*(e) = \bigcup_{k\in u} k(e)$, $\overline{u}$: $P_\omega(\rho) \longrightarrow P_\kappa(\lambda)$ by $\overline{u}(a) = \bigcap\{x \in C_{u^*}^{\kappa,\lambda} : \{0\} \cup a \subseteq x\}$, and $\widetilde{u}$: $P_\omega(\rho) \longrightarrow P_\kappa(\rho)$ by $\widetilde{u}(a) = \overline{u}(a) \cap \rho$.

Now fix f: $P_\omega(\rho) \longrightarrow P_\kappa(\rho)$. Select h: $P_\omega(\lambda) \longrightarrow P_\kappa(\rho)$ with $f \subseteq h$, and $K \in P_\kappa(\mathcal{K}) \setminus \{\emptyset\}$ with $\bigcap_{k\in K} C_k^{\kappa,\lambda} \subseteq C_h^{\kappa,\lambda}$. Set

$$B = \{b \in P_\kappa(\rho) : \forall u \in P_\omega(K) \setminus \{\emptyset\}(b \in C_{\widetilde{u}}^{\kappa,\rho})\}.$$

Now let $b \in B$. Put

$$y = b \cup \bigcup\{\overline{u}(a) : a \in P_\omega(b) \text{ and } u \in P_\omega(K) \setminus \{\emptyset\}\}.$$

We have $y \cap \rho = b$ since given $a \in P_\omega(b)$ and $u \in P_\omega(K) \setminus \{\emptyset\}$, $\overline{u}(a) \cap \rho = \widetilde{u}(a) \subseteq b$.

We claim that $y \in \bigcap_{k\in K} C_k^{\kappa,\lambda}$. To prove the claim, fix $k \in K$ and $e \in P_\omega(y)$. If $e \subseteq \rho$, then

$$k(e) \subseteq \{k\}^*(e) \subseteq \overline{\{k\}}(e) \subseteq y.$$

Now suppose $e \setminus \rho \neq \emptyset$. Let $e \setminus \rho = \{\xi_i : i \leq m\}$. For $i \leq m$, pick $a_i \in P_\omega(b)$ and $u_i \in P_\omega(K) \setminus \{\emptyset\}$ so that $\xi_i \in \overline{u_i}(a_i)$. Set $u = \{k\} \cup \bigcup_{i\leq m} u_i$ and $a = (e \cap \rho) \cup \bigcup_{i\leq m} a_i$. Since $e \cap \rho \subseteq a \subseteq \overline{u}(a)$ and $\overline{u_i}(a_i) \subseteq \overline{u}(a)$ for every $i \leq m$, we get $e \subseteq \overline{u}(a)$. Consequently,

$$k(e) \subseteq u^*(e) \subseteq \overline{u}(a) \subseteq y.$$

This completes the proof of the claim.

From the claim we obtain $y \in C_h^{\kappa,\lambda}$. Therefore, for every $d \in P_\omega(b)$,

$$f(d) = h(d) \subseteq y \cap \rho = b.$$

This yields $b \in C_f^{\kappa,\rho}$. So $B \subseteq C_f^{\kappa,\rho}$. It easily follows that $\overline{\mathrm{cof}}(NS_{\kappa,\rho}) \leq |\mathcal{K}|$. ∎

PROPOSITION 3.6: *Assume $\overline{\theta} \leq cf(\lambda) < \kappa$, and let ν be a cardinal with $\kappa \leq \nu < \lambda$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}) \leq \sup_{\nu\leq\rho<\lambda} \overline{\mathrm{cof}}(NS_{\kappa,\rho}^{[\rho]^{<\theta}})$.*

Proof: Set $\tau = cf(\lambda)$. Let $< \lambda_\gamma : \gamma < \tau >$ be a strictly increasing sequence of cardinals greater than or equal to ν such that $\lambda = \bigcup_{\gamma<\tau} \lambda_\gamma$. Given $\gamma < \tau$, select a collection $\mathcal{H}_\gamma$ of functions from $P_{\overline{\theta}\cdot 3}(\lambda_\gamma)$ to $P_\kappa(\lambda_\gamma)$ so that $|\mathcal{H}_\gamma| = \overline{\mathrm{cof}}(NS_{\kappa,\lambda_\gamma}^{[\lambda_\gamma]^{<\theta}})$ and for every k: $P_{\overline{\theta}\cdot 3}(\lambda_\gamma) \longrightarrow P_\kappa(\lambda_\gamma)$, there is $H \in P_\kappa(\mathcal{H}_\gamma) \setminus \{\emptyset\}$ with $\bigcap_{h\in H} C_h^{\kappa,\lambda_\gamma} \subseteq C_k^{\kappa,\lambda_\gamma}$. For $h \in \mathcal{H}_\gamma$, define h': $P_{\overline{\theta}\cdot 3}(\lambda) \longrightarrow P_\kappa(\lambda_\gamma)$ by $h'(d) = h(d \cap \lambda_\gamma)$. Notice that $a \cap \lambda_\gamma \in C_h^{\kappa,\lambda_\gamma}$ for every $a \in C_{h'}^{\kappa,\lambda}$.

Now fix f: $P_{\overline{\theta}\cdot 3}(\lambda) \longrightarrow P_\kappa(\lambda)$. For $\gamma < \tau$, define k_γ: $P_{\overline{\theta}\cdot 3}(\lambda_\gamma) \longrightarrow P_\kappa(\lambda_\gamma)$ by $k_\gamma(e) = f(e) \cap \lambda_\gamma$, and pick $H_\gamma \in P_\kappa(\mathcal{H}_\gamma) \setminus \{\emptyset\}$ with $\bigcap_{h\in H_\gamma} C_h^{\kappa,\lambda_\gamma} \subseteq C_{k_\gamma}^{\kappa,\lambda_\gamma}$. Let $a \in \bigcap_{\gamma<\tau} \bigcap_{h\in H_\gamma} C_{h'}^{\kappa,\lambda}$ and $e \in P_{|a\cap(\overline{\theta}\cdot 3)|}(a)$. There is $\xi < \tau$ such that $e \subseteq \lambda_\xi$. For $\xi \leq \gamma < \tau$, we have $a \cap \lambda_\gamma \in C_{k_\gamma}^{\kappa,\lambda_\gamma}$, hence $k_\gamma(e) \subseteq a \cap \lambda_\gamma$. Therefore, $f(e) \subseteq a$. So $\bigcap_{\gamma<\tau} \bigcap_{h\in H_\gamma} C_{h'}^{\kappa,\gamma} \subseteq C_f^{\kappa,\lambda}$. Now we can conclude that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}) \leq |\bigcup_{\gamma<\tau} \mathcal{H}_\gamma|$. ∎

We will see (Propositions 4.3 and 7.7 (ii)) that if λ is a strong limit cardinal, and $cf(\lambda) < \overline{\theta}$ or $cf(\lambda) \geq \kappa$, then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}) > \sup_{\kappa\leq\rho<\lambda} \overline{\mathrm{cof}}(NS_{\kappa,\rho}^{[\rho]^{<\theta}})$.

COROLLARY 3.7: *Assume $cf(\lambda) < \kappa$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) = \sup_{\kappa\leq\rho<\lambda} \overline{\mathrm{cof}}(NS_{\kappa,\rho})$.*

Proof: Use Proposition 3.5. ∎

COROLLARY 3.8: *Let μ be a singular strong limit cardinal, ν be a cardinal with $\nu > \mu$, and τ be a cardinal with $2 \leq \tau \leq cf(\mu)$. Then there exists $\eta < \mu$ such that $\overline{\mathrm{cof}}(NS_{\chi,\nu}^{[\mu]^{<\tau}}) = \nu$ for every regular uncountable cardinal χ with $\eta \leq \chi < \mu$.*

Proof: By a result of Shelah [9], there is a cardinal σ such that $2 \leq \sigma < \mu$ and $\mathrm{cov}(\nu, \mu, \mu, \sigma) = \nu$. Now let χ be any regular uncountable cardinal with $\sigma \leq \chi$ and $cf(\mu) < \chi < \mu$. By Propositions 1.2 and 1.3,

$$\nu \leq \mathrm{cov}(\nu, \mu^+, \mu^+, \chi) \leq \mathrm{cov}(\nu, \mu, \mu, \chi) \leq \mathrm{cov}(\nu, \mu, \mu, \sigma),$$

hence $\mathrm{cov}(\nu, \mu^+, \mu^+, \chi) = \nu$. From Proposition 3.6 we can infer that $\overline{\mathrm{cof}}(NS_{\chi,\mu}^{[\mu]^{<\tau}}) \leq \mu$. Hence by Proposition 3.1 $\overline{\mathrm{cof}}(NS_{\chi,\nu}^{[\mu]^{<\tau}}) = \nu$. ∎

4. $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A)$

Our aim in this section is to evaluate $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A)$ for A in $(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$. Let us first consider a few cases when $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A)$ does not depend on A.

PROPOSITION 4.1:

(i) *Assume that* $\lambda = \sigma^+$, $\kappa \le \delta$, $\sigma = \sigma^{<\kappa}$ *and* $\sigma^{|\delta|^{<\overline{\theta}}} \le \lambda$. *Then for every* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda$.

(ii) *Assume that* $\kappa \le \delta$ *and* λ *is a limit cardinal such that* $cf(\lambda) > |\delta|^{<\overline{\theta}}$. *Assume further that* $\tau^{|\delta|^{<\overline{\theta}}} < \lambda$ *for every cardinal* $\tau < \lambda$. *Then for every* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda$.

(iii) *Assume that* $cf(\lambda) < \kappa \le \delta$, *and* $\tau^{(|\delta|^{<\overline{\theta}})} < \lambda$ *for every cardinal* $\tau < \lambda$. *Then for every* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda$ *and* $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda^{cf(\lambda)}$.

(iv) *Assume that* λ *is a strong limit cardinal and* $\overline{\theta} \le cf(\lambda) < \kappa$. *Then for every* $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^+$, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) = \lambda$ *and* $\mathrm{cof}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) = \lambda^{cf(\lambda)}$.

Proof:

(i) and (ii) Let $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$. Then $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda$ since $\lambda \le \mathfrak{d}_{\kappa,\lambda}^{|\delta|^{<\overline{\theta}}} \le \lambda^{(|\delta|^{<\overline{\theta}})} = \lambda$. Furthermore, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda$ since $u(\kappa,\rho) < \lambda$ for every cardinal ρ with $\kappa \le \rho < \lambda$.

(iii) Let $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$. Since $\delta \ge \kappa$ and $2^{(|\delta|^{<\overline{\theta}})} < \lambda$, we get $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \mathfrak{d}_{\kappa,\lambda}^{|\delta|^{<\overline{\theta}}} = \lambda^{(|\delta|^{<\overline{\theta}})}$. Furthermore, since $cf(\lambda) < |\delta|^{<\overline{\theta}}$ and $\tau^{(|\delta|^{<\overline{\theta}})} < \lambda$ for every cardinal $\tau < \lambda$, we have $\lambda^{(|\delta|^{<\overline{\theta}})} = \lambda^{cf(\lambda)}$. With Corollary 3.2 and Corollary 3.3 we obtain

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) \le \sup_{|\delta|<\tau<\lambda} \mathrm{cof}(NS_{\kappa,\tau}^{[|\delta|]^{<\theta}}) \le \sup_{|\delta|<\tau<\lambda} \tau^{(|\delta|^{<\overline{\theta}})} \le \lambda.$$

Finally, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) \ge \lambda$ since $u(\kappa,\rho) < \lambda^{cf(\lambda)}$ for every cardinal ρ with $\kappa \le \rho < \lambda$.

(iv) Let $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^+$. Then $\mathrm{cof}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) = \mathfrak{d}_{\kappa,\lambda}^{\lambda^{<\overline{\theta}}} = \mathfrak{d}_{\kappa,\lambda}^{\lambda} \le (\lambda^{\kappa})^{\lambda} = \lambda^{cf(\lambda)}$. On the other hand, since $\lambda \ge 2^{<\kappa}$, we have $\mathfrak{d}_{\kappa,\lambda}^{\lambda} = \mathfrak{d}_{\kappa,\lambda^{<\kappa}}^{\lambda} \ge \lambda^{<\kappa} = \lambda^{cf(\lambda)}$. Proposition 3.6 implies

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) \le \sup_{\kappa\le\tau<\lambda} \mathrm{cof}(NS_{\kappa,\tau}^{[\tau]^{<\theta}}) \le \sup_{\kappa\le\tau<\lambda} \tau^{(\tau^{<\overline{\theta}})} \le \lambda.$$

Finally, $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) \geq \lambda$ since $u(\kappa,\rho) < \lambda^{cf(\lambda)}$ for every cardinal ρ with $\kappa \leq \rho < \lambda$. ∎

Our assumptions are sufficient but not necessary. To see this, suppose that GCH holds, $\delta \geq \kappa$ and either $\delta < \lambda = \sigma^+$ and $cf(\sigma) \geq \kappa$, or $\delta < \lambda$ and λ is a limit cardinal with $cf(\lambda) > |\delta|$, or λ is a limit cardinal with $cf(\lambda) < \kappa$. Let $\tau > \lambda$ be a regular cardinal, and P be the notion of forcing for adding τ Cohen reals. For each cardinal $\mu > 0$ and each cardinal $\rho \geq \kappa$, we have $(\mathfrak{d}_{\kappa,\rho}^{\mu})^{V^P} \leq (\mathfrak{d}_{\kappa,\rho}^{\mu})^{V^P}$ by a result of [7]. It follows that in V^P, $\overline{\text{cof}}(NS_{\kappa,\lambda}^{\delta}|A) = \lambda$ for every $A \in (NS_{\kappa,\lambda}^{\delta})^+$.

Proposition 4.2: *Let μ be a cardinal such that $\overline{\theta} \leq cf(\mu) < \kappa < \mu \leq \lambda$. Then $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}|A) \leq \sup_{\kappa\leq\tau<\mu} \overline{\text{cof}}(NS_{\kappa,\lambda}^{[\tau]^{<\theta}}|A)$ for all $A \in (NS_{\kappa,\lambda}^{[\mu]^{<\theta}})^+$.*

Proof: Fix $A \in (NS_{\kappa,\lambda}^{[\mu]^{<\theta}})^+$. Let $< \mu_\xi : \xi < cf(\mu) >$ be a strictly increasing sequence of cardinals greater than κ such that $\mu = \sup_{\xi<cf(\mu)} \mu_\xi$. Given $f: P_{\overline{\theta}\cdot 3}(\mu) \longrightarrow P_\kappa(\lambda)$, set $f_\xi = f \restriction P_{\overline{\theta}\cdot 3}(\mu_\xi)$ for every $\xi < cf(\mu)$. Then

$$\left\{a \in \bigcap_{\xi<cf(\mu)} (A \cap C_{f_\xi}^{\kappa,\lambda}) : \overline{\theta} \subseteq a\right\} \subseteq A \cap C_f^{\kappa,\lambda}.$$

So we get

$$\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}|A) \leq cf(\mu) \cdot (\sup_{\xi<cf(\mu)} \overline{\text{cof}}(NS_{\kappa,\lambda}^{[\mu_\xi]^{<\theta}}|A)).$$

The desired result follows, since $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\mu]^{<\theta}}|A) > cf(\mu)$ by Proposition 2.1. ∎

Next we investigate lower bounds for $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A)$.

Proposition 4.3: *Let μ be a cardinal with $\kappa \leq \mu \leq \lambda$, and J be a $[\mu]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. Then*

(i) *$cf(\overline{\text{cof}}(J)) < \kappa$ or $cf(\overline{\text{cof}}(J)) > \mu^{<\overline{\theta}}$.*
(ii) *Assume $\mu^{<\overline{\theta}} = \mu^\kappa$. Then $\overline{\text{cof}}(J) > \mu^{<\overline{\theta}}$.*

Proof:

(i) Suppose that $\kappa \leq \rho \leq \mu^{<\overline{\theta}}$, where $\rho = cf(\overline{\text{cof}}(J))$. Pick $E \subseteq P_{\overline{\theta}\cdot 3}(\mu)$ with $|E| = \rho$, and let $E = \{e_\alpha : \alpha < \rho\}$. Select $X_\alpha \subseteq J$ for $\alpha < \rho$ so that
   (i) $|X_\alpha| < \overline{\text{cof}}(J)$.
   (ii) $X_\beta \subseteq X_\alpha$ for all $\beta < \alpha$.

(iii) For every $B \in J$, there is $S \in P_\kappa(\bigcup_{\alpha<\rho} X_\alpha)$ with $B \subseteq \cup S$.

Given $\alpha < \rho$, set $d_\alpha = e_\alpha \cup (\overline{\theta} \cdot 3)$ if $\overline{\theta} < \kappa$, and $d_\alpha = e_\alpha \cup |e_\alpha|^+$ otherwise. Let $Y_\alpha = \{A \cup (P_\kappa(\lambda) \setminus \widehat{d_\alpha}) : A \in X_\alpha\}$, and pick $B_\alpha \in J$ so that $B_\alpha \not\subseteq \cup T$ for every $T \in P_\kappa(Y_\alpha)$. Now put

$$B = \bigcup_{\alpha<\rho} (B_\alpha \cap \{a \in P_\kappa(\lambda) : e_\alpha \in P_{|a\cap(\overline{\theta}\cdot 3)|}(a)\}).$$

Since $B \in J$, we can find $\alpha < \rho$ and $S \in P_\kappa(X_\alpha)$ so that $B \subseteq \cup S$. Setting $T = \{A \cup (P_\kappa(\lambda) \setminus \widehat{d_\alpha}) : A \in S\}$, we have $B_\alpha \subseteq B \cup (P_\kappa(\lambda) \setminus \widehat{d_\alpha}) \subseteq \cup T$. This yields the desired contradiction.

(ii) We have $\text{cof}(J) \geq \eth^{\mu^{<\overline{\theta}}}_{\kappa,\lambda} > \mu^{<\overline{\theta}} = \mu^\kappa$. From $u(\kappa, \mu^\kappa) = \mu^\kappa$ we can conclude that $\overline{\text{cof}}(J) > \mu^\kappa$. ∎

In particular, if $\delta > \kappa$ and $|\delta|$ is a strong limit cardinal with $cf(|\delta|) < \overline{\theta}$, then $\overline{\text{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda} | A) > 2^{|\delta|}$ for every $A \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+$.

Definition: Given $f: P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$ and $E \subseteq \lambda$, $\Gamma_f(E)$ is defined as follows. Set $\rho = \overline{\theta} \cdot \aleph_0$ if $\overline{\theta} \cdot \aleph_0$ is a regular cardinal, and $\rho = (\overline{\theta} \cdot \aleph_0)^+$ otherwise. Define $E_\alpha \subseteq \lambda$ for $\alpha < \rho$ by:

(a) $E_0 = E$.

(b) $E_{\alpha+1} = E_\alpha \cup (\cup f[P_{\overline{\theta}\cdot 3}(E_\alpha \cap \delta)])$.

(c) $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$ if α is an infinite limit ordinal.

Then let $\Gamma_f(E) = \bigcup_{\alpha<\rho} E_\alpha$.

It is simple to see that

$$\Gamma_f(E) = \bigcap\{D : E \subseteq D \subseteq \lambda \text{ and } \forall e \in P_{\overline{\theta}\cdot 3}(D \cap \delta)(f(e) \subseteq D)\}.$$

LEMMA 4.4: *Let δ' be an ordinal with $1 \leq \delta' \leq \lambda$, and θ' be a cardinal with $2 \leq \theta' \leq \kappa$. Further, let σ be a cardinal such that $\sigma > \kappa \cdot |\delta|^{<\overline{\theta}}$ and $\Gamma_f(E) \neq \lambda$ for all $E \in P_\sigma(\lambda)$ and $f: P_{\overline{\theta'}\cdot 3}(\delta') \longrightarrow P_\kappa(\lambda)$. Then $\overline{\text{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda} | A) \geq \sigma$ for every $A \in (NS^{[\delta']^{<\theta'}}_{\kappa,\lambda})^*$.*

Proof: Let $A \in (NS^{[\delta']^{<\theta'}}_{\kappa,\lambda})^*$ and $B_\alpha \in NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \mid A$ for $\alpha < \mu$, where μ is a cardinal with $0 < \mu < \sigma$. Pick $f: P_{\theta'\cdot 3}(\delta') \longrightarrow P_\kappa(\lambda)$ with $C^{\kappa,\lambda}_f \subseteq A$, and for $\alpha < \mu$, $g_\alpha: P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$ with $(B_\alpha \cap A) \cap C^{\kappa,\lambda}_{g_\alpha} = \emptyset$. Set $E = \kappa \cup \bigcup_{\alpha<\mu}(\cup \text{ran}(g_\alpha))$. Since $|E| < \sigma$, we can find $\zeta \in \lambda$ so that $\zeta \notin \Gamma_f(E)$.

Now let $x \in P_\kappa(\mu) \setminus \{\emptyset\}$. Define b as follows. First suppose that $\overline{\theta} < \kappa$ and $\overline{\theta'} < \kappa$. Set $\rho = (\overline{\theta} \cdot \aleph_0) \cup (\overline{\theta'} \cdot \aleph_0)$ if $(\overline{\theta} \cdot \aleph_0) \cup (\overline{\theta'} \cdot \aleph_0)$ is a regular cardinal, and $\rho = ((\overline{\theta} \cdot \aleph_0) \cup (\overline{\theta'} \cdot \aleph_0))^+$ otherwise. Define a_β for $\beta < \rho$ by:

(a) $a_0 = (\overline{\theta} \cdot 3) \cup (\overline{\theta'} \cdot 3)$.

(b) $a_{\beta+1} = a_\beta \cup v \cup \omega$, where $v = \bigcup\{f(d) : d \in P_{\overline{\theta'}\cdot 3}(a_\beta \cap \delta')\}$ and $w = \bigcup\{g_\alpha(e) : \alpha \in x \text{ and } e \in P_{\overline{\theta}\cdot 3}(a_\beta \cap \delta)\}$.

(c) $a_\beta = \bigcup_{\xi<\beta} a_\xi$ if β is an infinite limit ordinal.

Now let $b = \bigcup_{\beta<\rho} a_\beta$. Next suppose that $\overline{\theta} = \kappa$ or $\overline{\theta'} = \kappa$. Define s_β and γ_β for $\beta < \kappa$ by:

(i) If $\overline{\theta} < \kappa$, $s_0 = \overline{\theta} \cdot 3$. If $\overline{\theta'} < \kappa$, $s_0 = \overline{\theta'} \cdot 3$. If $\overline{\theta} = \overline{\theta'} = \kappa$, $s_0 = \{0\}$.

(ii) $\gamma_\beta = \bigcup(d_\beta \cap \kappa)$.

(iii) $s_{\beta+1} = s_\beta \cup (\gamma_\beta + 2) \cup y \cup z$, where $y = \bigcup\{f(d) : d \in P_{|s_\beta \cap (\overline{\theta'}\cdot 3)|}(d_\beta \cap \delta')\}$ and $z = \bigcup\{g_\alpha(e) : \alpha \in x \text{ and } e \in P_{|s_\beta \cap (\overline{\theta}\cdot 3)|}(s_\beta \cap \delta)\}$.

(iv) $s_\beta = \bigcup_{\xi<\beta} s_\xi$ if β is an infinite limit ordinal.

Select a regular infinite cardinal $\tau < \kappa$ so that

(0) $\gamma_\tau = \tau$.

(1) If $\overline{\theta} < \kappa$, then $\overline{\theta} \leq \tau$.

(2) If $\overline{\theta'} < \kappa$, then $\overline{\theta'} \leq \tau$.

In any case we have $b \in C_f^{\kappa,\lambda} \cap \bigcap_{\alpha\in x} C_{g_\alpha}^{\kappa,\lambda}$. Moreover, $\zeta \notin b$ since $b \subseteq \Gamma_f(E)$. Hence $P_\kappa(\lambda) - \widehat{\{\zeta\}} \not\subseteq \bigcup_{\alpha\in x} B_\alpha$. ∎

PROPOSITION 4.5: *Let δ' be an ordinal with $1 \leq \delta' \leq \lambda$, and θ' be a cardinal with $2 \leq \theta' \leq \kappa$. Assume that $\lambda > \kappa \cdot |\delta|^{<\overline{\theta}} \cdot |\delta'|^{<\overline{\theta'}}$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) \geq \lambda$ for every $A \in (NS_{\kappa,\lambda}^{[\delta']^{<\theta'}})^*$.*

Proof: The result follows from Lemma 4.4 since $|\Gamma_f(E)| < \lambda$ for all $E \in P_\kappa(\lambda)$ and $f\colon P_{\overline{\theta'}\cdot 3}(\delta') \longrightarrow P_\kappa(\lambda)$. ∎

COROLLARY 4.6: *Let δ' be an ordinal with $1 \leq \delta' \leq \lambda$, and θ' be a cardinal with $2 \leq \theta' \leq \kappa$. Assume that $\kappa \leq \delta$, $\kappa \leq cf(\lambda) \leq |\delta|^{<\overline{\theta}} < \lambda$ and $|\delta'|^{<\overline{\theta'}} < \lambda$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) > \lambda$ for every $A \in (NS_{\kappa,\lambda}^{[\delta']^{<\theta'}})^*$.*

Proof: Use Proposition 4.3. ∎

PROPOSITION 4.7: *Let θ' be a cardinal with $2 \leq \theta' \leq \kappa$, and σ be the least cardinal τ such that $\tau^{<\overline{\theta'}} \geq \kappa$. Assume that $\kappa < \lambda$ and $|\delta|^{<\overline{\theta}} < \sigma$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) \geq \sigma$ for every $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta'}})^*$.*

Proof: Suppose that there exists a $[\lambda]^{<\theta'}$-normal ideal on $P_\kappa(\lambda)$. Then $\sigma > \kappa$, and $(\nu^{<\overline{\theta'}})^{<\overline{\theta'}} = \nu^{<\overline{\theta'}}$ for every cardinal $\nu \geq \kappa$. Hence for each $f\colon P_{\overline{\theta'}\cdot 3}(\lambda) \longrightarrow$

$P_\kappa(\lambda)$ and each $E \in P_\sigma(\lambda) \setminus \{\emptyset\}$, we get $|\Gamma_f(E)| \le (\kappa \cdot |E|)^{<\overline{\theta'}} < \lambda$. Now apply Lemma 4.4. ∎

In particular, if $\lambda > \kappa \cdot |\delta|^{<\overline{\theta}}$ and $A \in NS^*_{\kappa,\lambda}$, then $\overline{\mathrm{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A) \ge \lambda$. If in addition $\delta \ge \kappa$ and $\kappa \le cf(\lambda) \le |\delta|^{<\overline{\theta}}$, then $\overline{\mathrm{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A) > \lambda$ — see the next result.

Corollary 4.8: *Let θ' be a cardinal with $2 \le \theta' \le \kappa$. Assume that $\kappa \le \delta$, $\kappa \le cf(\lambda) \le |\delta|^{<\overline{\theta}} < \lambda$, and $\mu^{<\overline{\theta'}} < \lambda$ for every cardinal $\mu < \lambda$. Then $\overline{\mathrm{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A) > \lambda$ for all $A \in (NS^{[\lambda]^{<\theta'}}_{\kappa,\lambda})^*$.*

Proof: Use Proposition 4.3. ∎

5. $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A)$

Proposition 5.1: $\overline{\mathrm{cof}}(I_{\kappa,\lambda}) = \lambda$.

Proof: For each $B \in I_{\kappa,\lambda}$, there is $a \in P_\kappa(\lambda)$ such that $B \subseteq \bigcup_{\alpha\in a}(P_\kappa(\lambda)\setminus\widehat{\{\alpha\}})$. From this we get at once $\overline{\mathrm{cof}}(I_{\kappa,\lambda}) \le \lambda$. The reverse inequality is immediate from the remark that given fewer than λ many sets in $P_\kappa(\lambda)$ their union is a proper subset of λ. ∎

The rest of the section deals with the question of whether there exists A such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) < \lambda$.

Proposition 5.2:
- (i) *If $\lambda \le \kappa^{+\omega}$, then $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) = \lambda$ for every $A \in I^+_{\kappa,\lambda}$.*
- (ii) *If $|\delta|^{<\overline{\theta}} < \lambda$, then $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) = \lambda$ for every $A \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^*$.*
- (iii) *Let σ be the least cardinal τ such that $\tau^{<\overline{\theta}} \ge \lambda$. Then $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \ge \sigma$ for every $A \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^*$.*

Proof: By Propositions 5.1, 2.2, 4.5 and 4.7. ∎

Proposition 5.3: *Let σ be a cardinal with $\kappa < \sigma < \lambda$. Setting $\theta = (cf(\sigma))^+$, assume that $\theta < \kappa$, there exists a $[\sigma]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, and σ is the least cardinal τ such that $\tau^{<\theta} \ge \lambda$. Then $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) = \sigma$ for some $A \in (NS^{[\sigma]^{<\theta}}_{\kappa,\lambda})^*$.*

Proof: Pick a surjection $j : P_\theta(\sigma) \longrightarrow P_2(\lambda)$. Then given $\gamma \in \lambda$, we have

$$C_j^{\kappa,\lambda} \cap \widehat{\theta} \cap \bigcap_{\alpha\in e} \widehat{\{\alpha\}} \subseteq \widehat{\{\gamma\}}$$

for any $e \in P_\theta(\sigma)$ such that $j(e) = \{\gamma\}$. Consequently, $\overline{\text{cof}}(I_{\kappa,\lambda}|C_j^{\kappa,\lambda}) \leq \sigma$. The reverse inequality holds by Proposition 5.2 (iii). ∎

If $\lambda = \kappa$, then by Proposition 2.1, $\overline{\text{cof}}(I_{\kappa,\lambda}|A) = \lambda$ for all A. Now suppose $\lambda > \kappa$. If $u(\kappa,\tau) < \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$ (under GCH, this assumption is equivalent to: λ is either a limit cardinal or the successor of a cardinal of cofinality greater than or equal to κ), then by Proposition 2.1, $\overline{\text{cof}}(I_{\kappa,\lambda}|A) = \lambda$ for all A. On the other hand, if there exist two cardinals $\tau < \kappa$ and $\rho < \lambda$ such that $\rho^{<\tau} \geq \lambda$ and $\mu^{<\tau} < \kappa$ for every cardinal $\mu < \kappa$ (under GCH, this means that λ is the successor of a cardinal σ of cofinality less than κ and κ is not the successor of a cardinal of cofinality less than $cf(\sigma)$), then by Proposition 5.3, $\overline{\text{cof}}(I_{\kappa,\lambda}|A) < \lambda$ for some A. We will see that the same conclusion follows from the hypothesis that λ is the successor of a cardinal σ such that $cf(\sigma) < \kappa$ and $\square_\sigma^*$ holds. On the other hand, the conclusion fails if $\lambda = \sigma^+$, $\kappa = \nu^+$, $\sigma^\nu = \sigma$, $\nu^{<cf(\nu)} = \nu$ and $\lambda \longrightarrow [\kappa]^2_{\sigma,<\kappa}$. Thus, the assertion "$\overline{\text{cof}}(I_{\omega_1,\omega_{\omega+1}}|A) < \omega_{\omega+1}$ for some A" is consistent with ZFC, but so is (relative to a large cardinal) its negation. First, we reformulate our problem.

Definition: For two cardinals ρ and σ such that $2 \leq \rho \leq \kappa \leq \sigma$, $\mathcal{A}_{\kappa,\lambda}^{\sigma,\rho}$ asserts the existence of $y_\alpha \in P_\rho(\sigma)$ for $\alpha < \lambda$ such that $|\{\alpha < \lambda : y_\alpha \subseteq d\}| < \kappa$ for all $d \in P_\kappa(\sigma)$.

LEMMA 5.4: *Let ρ and σ be two cardinals such that $2 \leq \rho \leq \kappa \leq \sigma$, and let $y_\alpha \in P_\rho(\sigma)$ for $\alpha < \lambda$ be such that $|\{\alpha < \lambda : y_\alpha \subseteq d\}| < \kappa$ for every $d \in P_\kappa(\sigma)$. Then $|\{\alpha < \lambda : y_\alpha \subseteq x\}| \leq \kappa$ for every $x \in P_{\kappa^+}(\sigma)$.*

Proof: Suppose, to get a contradiction, that there is $e \subseteq \lambda$ such that $|e| = \kappa^+$ and $|\bigcup_{\alpha\in e} y_\alpha| < \kappa^+$. Then $|\bigcup_{\alpha\in e} y_\alpha| = \kappa$. Select a bijection $j\colon \kappa \longrightarrow \bigcup_{\alpha\in e} y_\alpha$. For $\alpha \in e$, let ξ_α be the least $\beta < \kappa$ such that $y_\alpha \subseteq j[\beta]$. Pick $e' \subseteq e$ and $\xi < \kappa$ so that $|e'| = \kappa^+$ and $\xi_\alpha = \xi$ for all $\alpha \in e'$. Then $|\bigcup_{\alpha\in e'} y_\alpha| < \kappa$, a contradiction. ∎

PROPOSITION 5.5: *Let ρ and σ be two cardinals such that $2 \leq \rho \leq \kappa < \sigma$. Then $\mathcal{A}_{\kappa,\lambda}^{\sigma,\rho}$ implies $\mathcal{A}_{\kappa^+,\lambda}^{\sigma,\rho}$.*

Proof: This is an immediate consequence of Lemma 5.4. ∎

Definition: Given a cardinal $\sigma \geq \kappa$, $\mathcal{A}_{\kappa,\lambda}^{\sigma}$ stands for $\mathcal{A}_{\kappa,\lambda}^{\sigma,\kappa}$.

PROPOSITION 5.6:

(i) *Let σ be a cardinal with $\kappa \le \sigma$. Assume that $\mathcal{A}^{\sigma}_{\kappa,\lambda}$ holds, $\delta < \lambda$ and there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. Then there is $D \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) \le \sigma$.*

(ii) *Let σ be a cardinal with $\kappa \le \sigma$, and ρ be a regular infinite cardinal with $\rho < \kappa$. Assume that $\mathcal{A}^{\sigma,\rho}_{\kappa,\lambda}$ holds and there exists a $[\lambda]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. Then there is $D \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) \le \sigma$.*

Proof:

(i) The result is trivial in case $\sigma \ge \lambda$. Now assume $\sigma < \lambda$. Select $y_\alpha \in P_\kappa(\sigma)$ for $\alpha \in \lambda \setminus \delta$ so that $|\{\alpha \in \lambda \setminus \delta : y_\alpha \subseteq d\}| < \kappa$ for every $d \in P_\kappa(\sigma)$. Let D be the set of all $a \in P_\kappa(\lambda)$ such that $\{\alpha \in \lambda \setminus \delta : y_\alpha \subseteq a\} \subseteq a$. Then $D \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+$, since given $f\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$, we have

$$b \bigcup \{\alpha \in \lambda \setminus \delta : y_\alpha \subseteq b\} \in D \cap C^{\kappa,\lambda}_f$$

for any $b \in C^{\kappa,\lambda}_f$. Furthermore, $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) \le \sigma$ since given $c \in P_\kappa(\lambda)$, we have $D \cap \widehat{c'} \subseteq \widehat{c}$, where $c' = (c \cap \sigma) \cup \bigcup_{\alpha \in c \setminus \sigma} y_\alpha$.

(ii) Let us assume that $\sigma < \lambda$, since otherwise the result is trivial. Select $y_\alpha \in P_\rho(\sigma)$ for $\alpha < \lambda$ so that $|\{\alpha < \lambda : y_\alpha \subseteq d\}| < \kappa$ for every $d \in P_\kappa(\sigma)$. Let D be the set of all $a \in P_\kappa(\lambda)$ such that $\{\alpha \in \lambda : y_\alpha \subseteq a\} \subseteq a$. To prove that $D \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$, fix $f\colon P_{\overline{\theta}\cdot 3}(\lambda) \longrightarrow P_\kappa(\lambda)$. First suppose $\overline{\theta} < \kappa$. Pick a regular cardinal χ so that $\rho \cdot \overline{\theta} \le \chi < \kappa$. Now define a_β for $\beta \le \chi$ by:

(a) $a_0 = \overline{\theta} \cdot 3$.

(b) $a_{\beta+1} = a_\beta \bigcup \{\alpha \in \lambda : y_\alpha \subseteq a\} \cup \bigcup_{e \in P_{\overline{\theta}\cdot 3}(a_\beta)} f(e)$.

(c) $a_\beta = \bigcup_{\gamma<\beta} a_\gamma$ if β is an infinite limit ordinal.

Then we have $a_\chi \in D \cap C^{\kappa,\lambda}_f$. Next suppose $\overline{\theta} = \kappa$. Define b_β and γ_β for $\beta < \kappa$ by:

(0) $b_0 = \omega$.

(1) $\gamma_\beta = \cup(b_\beta \cap \kappa)$.

(2) $b_{\beta+1} = b_\beta \cup (\gamma_\beta + 2) \cup |b_\beta|^+ \cup \{\alpha \in \lambda : y_\alpha \subseteq b_\beta\} \cup \bigcup_{e \subseteq b_\beta} f(e)$.

(3) $b_\beta = \bigcup_{\xi<\beta} b_\xi$ if β is an infinite limit ordinal.

Now select a regular cardinal τ so that $\rho < \tau < \kappa$ and $\gamma_\tau = \tau$. Since $|b_\tau| = \tau = b_\tau \cap \tau$, we get $b_\tau \in D \cap C^{\kappa,\lambda}_f$.

Finally, to see that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) \le \sigma$, it suffices to observe that for any $c \in P_\kappa(\lambda)$, $D \cap \widehat{c'} \subseteq \widehat{c}$, where $c' = \bigcup_{\alpha \in c} y_\alpha$. ∎

PROPOSITION 5.7: *Given a cardinal $\sigma \geq \kappa$, the following are equivalent:*

(i) $\mathcal{A}^\sigma_{\kappa,\lambda}$ *holds.*

(ii) $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \leq \sigma$ *for some* $A \in I^+_{\kappa,\lambda}$.

(iii) *There is an ideal J on $P_\kappa(\lambda)$ such that* $\overline{\mathrm{cof}}(J) \leq \sigma$.

Proof:

(i) $\to$ (ii) By Proposition 5.6 (i).

(ii) $\to$ (iii) Trivial.

(iii) $\to$ (i) Let J be an ideal on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) \leq \sigma$. Select $D_\beta \in J^*$ for $\beta < \sigma$ so that for every $D \in J^*$, there is $x \in P_\kappa(\sigma) \setminus \{\emptyset\}$ with $\bigcap_{\beta\in x} D_\beta \subseteq D$. For $\alpha \in \lambda$, pick $y_\alpha \in P_\kappa(\sigma) \setminus \{\emptyset\}$ so that $\bigcap_{\beta\in y_\alpha} D_\beta \subseteq \widehat{\{\alpha\}}$. Now let $d \in P_\kappa(\sigma) \setminus \{\emptyset\}$. Then $\{\alpha < \lambda : y_\alpha \subseteq d\} \subseteq c$ for any $c \in \bigcap_{\beta\in d} D_\beta$, hence $|\{\alpha < \lambda : y_\alpha \subseteq d\}| < \kappa$. ∎

COROLLARY 5.8: *Let σ be a cardinal such that $\kappa \leq \sigma \leq \lambda$ and $\mathcal{A}^\sigma_{\kappa,\lambda}$ holds. Then $u(\kappa,\sigma) = u(\kappa,\lambda)$.*

Proof: By Proposition 5.7, there is $A \in I^+_{\kappa,\lambda}$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \leq \sigma$. Then we get

$$u(\kappa,\lambda) = \mathrm{cof}(I_{\kappa,\lambda}|A) \leq u(\kappa, \overline{\mathrm{cof}}(I_{\kappa,\lambda}|A)) \leq u(\kappa,\sigma) \leq u(\kappa,\lambda). \quad ∎$$

We now consider the question of whether there exists $D \in NS^+_{\kappa,\lambda}$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) < \lambda$. Proposition 5.6 (ii) gives a positive answer in some cases, but it does not apply if, e.g., $\kappa = \omega_1$ and $\lambda = \omega_{\omega+1}$. To deal with such cases we introduce a (stronger) variant of $\mathcal{A}^\sigma_{\kappa,\lambda}$.

Definition: For two cardinals ρ and σ such that $2 \leq \rho \leq \kappa \leq \sigma$, $\mathcal{B}^{\sigma,\rho}_{\kappa,\lambda}$ asserts the existence of $y_\alpha \in P_\rho(\sigma)$ for $\alpha < \lambda$ such that for every nonempty $e \in P_{\kappa^+}(\lambda)$, there is a $< \kappa$-to-one function in $\prod_{\alpha\in e} y_\alpha$.

LEMMA 5.9: *Let ρ and σ be two cardinals such that $2 \leq \rho \leq \kappa \leq \sigma$, and let $y_\alpha \in P_\rho(\sigma)$ for $\alpha < \lambda$ be such that for every nonempty $e \in P_{\kappa^+}(\lambda)$, there is a $< \kappa$-to-one function in $\prod_{\alpha\in e} y_\alpha$. Then $|\{\alpha < \lambda : y_\alpha \subseteq d\}| < \kappa$ for every $d \in P_\kappa(\sigma)$.*

Proof: We have to show that $|\bigcup_{\alpha\in e} y_\alpha| = \kappa$ for every $e \subseteq \lambda$ with $|e| = \kappa$. Given such an e, select a $< \kappa$-to-one function $h \in \prod_{\alpha\in e} y_\alpha$. Define by induction $\xi_\beta \in e$ for $\beta < \kappa$ so that $h(\xi_\beta) \neq h(\xi_\gamma)$ for all $\gamma < \beta$. Then clearly $|\bigcup_{\beta<\kappa} y_{\xi_\beta}| = \kappa$.
∎

PROPOSITION 5.10: *Let ρ and σ be two cardinals such that $2 \leq \rho \leq \kappa \leq \sigma$. Then $\mathcal{B}^{\sigma,\rho}_{\kappa,\lambda}$ implies $\mathcal{A}^{\sigma,\rho}_{\kappa,\lambda}$.*

Proof: The result follows immediately from Lemma 5.9. ∎

Definition: Given a cardinal $\sigma \geq \kappa$, $\mathcal{B}^{\sigma}_{\kappa,\lambda}$ stands for $\mathcal{B}^{\sigma,\kappa}_{\kappa,\lambda}$.

PROPOSITION 5.11: *Let σ be a cardinal such that $\sigma \geq \kappa$ and $\mathcal{B}^{\sigma}_{\kappa,\lambda}$ holds. Assume that there exists a $[\lambda]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. Then there is $D \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) \leq \sigma$.*

Proof: Let us assume that $\sigma < \lambda$, since otherwise the result is trivial. Select $y_\alpha \in P_\kappa(\sigma)$ for $\alpha < \lambda$ so that for every nonempty $e \in P_{\kappa^+}(\lambda)$, there is a $< \kappa$-to-one function in $\prod_{\alpha\in e} y_\alpha$. Let D be the set of all $a \in P_\kappa(\lambda)$ such that $\{\alpha < \lambda : y_\alpha \subseteq a\} \subseteq a$. To prove that $D \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$, fix $f\colon P_{\bar{\theta}\cdot 3}(\lambda) \longrightarrow P_\kappa(\lambda)$. Define e_β for $\beta < \kappa$ by:

(a) $e_0 = \kappa$.

(b) $e_{\beta+1} = e_\beta \cup \{\alpha < \lambda : y_\alpha \subseteq e_\beta\} \cup \bigcup_{b\in P_{\bar{\theta}\cdot 3}(e_\beta)} f(b)$.

(c) $e_\beta = \bigcup_{\gamma<\beta} e_\gamma$ if β is an infinite limit ordinal.

Now set $E = \bigcup_{\beta<\kappa} e_\beta$. Then $|E| = \kappa$ and $\{\alpha < \lambda : y_\alpha \subseteq E\} \subseteq E \subseteq \lambda$. Moreover, $f(b) \subseteq E$ for every $b \in P_{\bar{\theta}\cdot 3}(E)$. Select a $< \kappa$-to-one $h \in \prod_{\alpha\in E} y_\alpha$, and let H be the set of all $a \in P_\kappa(\lambda)$ such that $h^{-1}(\{\xi\}) \subseteq a$ for every $\xi \in a \cap \mathrm{ran}(h)$. Clearly, $H \in (NS^{[\sigma]^{<\theta}}_{\kappa,\lambda})^*$. Pick $a \in \widehat{\{0\}} \cap H \cap C^{\kappa,\lambda}_f$. It is simple to see that $a \cap E \in C^{\kappa,\lambda}_f$. Now suppose that $\alpha \in \lambda$ is such that $y_\alpha \subseteq a \cap E$. Then we get $\alpha \in E$ and $h(\alpha) \in a \cap \mathrm{ran}(h)$. Since $a \in H$, we can conclude that $\alpha \in a$. Thus $a \cap E \in D$, hence $D \cap C^{\kappa,\lambda}_f \neq \emptyset$. Finally, if $c \in P_\kappa(\lambda)$, then $D \cap \widehat{c'} \subseteq \widehat{c}$, where $c' = \bigcup_{\alpha\in c} y_\alpha$. This yields $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) \leq \sigma$. ∎

6. $\mathcal{A}^{\sigma}_{\kappa,\lambda}$ and $\mathcal{B}^{\sigma}_{\kappa,\lambda}$

This section is concerned with the truth of $\mathcal{A}^{\sigma}_{\kappa,\lambda}$ and $\mathcal{B}^{\sigma}_{\kappa,\lambda}$.

Definition: Given a set A, we set $[A]^2 = \{a \subseteq A : |a| = 2\}$.

Definition: Given two cardinals χ and τ, $\lambda \longrightarrow [\kappa]^2_{\chi,<\tau}$ means that for every $F\colon [\lambda]^2 \longrightarrow \chi$, there is $A \subseteq \lambda$ such that $|A| = \kappa$ and $|\{F(a) : a \in [A]^2\}| < \tau$.

PROPOSITION 6.1: *Let μ be a singular limit cardinal such that $cf(\mu) < \kappa \le 2^{<\mu}$ and $\kappa \longrightarrow [\kappa]^2_{cf(\mu),<cf(\mu)}$. Then setting $\sigma = 2^{<\mu}$, $\rho = (cf(\mu))^+$ and $\lambda = 2^\mu$, $\mathcal{A}^{\sigma,\rho}_{\kappa,\lambda}$ holds.*

Proof: Select a strictly increasing sequence $< \mu_\gamma : \gamma < cf(\mu) >$ of infinite cardinals so that $\mu = \sup_{\gamma<cf(\mu)} \mu_\gamma$. Let Q be the set of all $X \subseteq \mu$ such that $\{\mu_\gamma : \gamma < cf(\mu)\} \subseteq X$. Pick a bijection j: $\bigcup_{\gamma<cf(\mu)} P(\mu_\gamma) \longrightarrow \sigma$. For $X \in Q$, let $y_X = \{j(X \cap \mu_\gamma) : \gamma < cf(\mu)\}$. Notice that $y_X \subseteq \sigma$ and $|y_X| = cf(\mu)$. Now fix $\mathfrak{X} \subseteq Q$ with $|\mathfrak{X}| = \kappa$. Define F: $[\mathfrak{X}]^2 \longrightarrow cf(\mu)$ by: $F(\{X, X'\})$ = the least $\gamma < cf(\mu)$ such that $X \cap \mu_\gamma \neq X' \cap \mu_\gamma$. Select $\mathcal{Y} \subseteq \mathfrak{X}$ and $\eta < cf(\mu)$ so that $|\mathcal{Y}| = \kappa$ and $F(w) \le \eta$ for all $w \in [\mathcal{Y}]^2$. Define k: $\mathcal{Y} \longrightarrow \bigcup_{X\in\mathcal{Y}} y_X$ by $k(X) = j(X \cap \mu_\eta)$. Then k is one-to-one, hence $|\bigcup_{X\in\mathfrak{X}} y_X| = \kappa$. Since $|Q| = \lambda$, we can conclude that $\mathcal{A}^{\sigma,\rho}_{\kappa,\lambda}$ holds. ∎

The following is due to Shelah (see Theorem 6.3 in Chapter II of [8]).

PROPOSITION 6.2: *Let ρ and σ be two cardinals such that $cf(\sigma) < \rho \le \kappa < \sigma < \lambda$. Assume that $u(\sigma^+, \lambda) < \mathrm{cov}(\sigma, \sigma, \rho, 2)$. Then $\mathcal{B}^{\sigma,\rho}_{\kappa,\lambda}$ holds.*

Proof: Select $B \in I^+_{\sigma^+,\lambda}$ so that $|B| = u(\sigma^+, \lambda)$. For $b \in B$, let $b = \bigcup_{\gamma<cf(\sigma)} d^b_\gamma$, where $|d^b_\gamma| < \sigma$ for every $\gamma < cf(\sigma)$. Pick $y_\alpha \in P_\rho(\sigma)$ for $\alpha < \lambda$ so that $y_\alpha \not\subseteq \bigcup_{\zeta\in\alpha\cap d^b_\gamma} y_\zeta$ for every $b \in B$ and every $\gamma < cf(\sigma)$. Now let $e \in P_{\sigma^+}(\lambda) \setminus \{\emptyset\}$. Select $b \in B$ so that $e \subseteq b$. Define g: $e \longrightarrow cf(\sigma)$ by: $g(\alpha)$ = the least $\gamma < cf(\sigma)$ such that $\alpha \in d^b_\gamma$. Define $h \in \prod_{\alpha\in e} y_\alpha$ so that $h(\alpha) \notin \bigcup_{\zeta\in\alpha\cap d^b_{g(\alpha)}} y_\zeta$ for $\alpha \in e$. Given $u \subseteq e$ with $|u| = (cf(\sigma))^+$, select $v \subseteq u$ so that $|v| = (cf(\sigma))^+$ and g is constant on v. Then h is one-to-one on v and therefore not constant on u. Thus h is $< (cf(\sigma))^+$-to-one. ∎

COROLLARY 6.3: *Let ρ and σ be two cardinals such that* (a) *$cf(\sigma) < cf(\rho)$,* (b) *$\rho \le \kappa$,* (c) *$\kappa \cdot 2^{<\rho} < \sigma$,* (d) *$\sigma^+ < \sigma^{<\rho}$, and* (e) *$u(\rho, \nu) < \sigma$ for every cardinal ν with $\rho \le \nu < \sigma$. Then $\mathcal{B}^{\sigma,\rho}_{\kappa,\sigma^+}$ holds.*

Proof: It is simple to see that $\sigma^{<\rho} = 2^{<\rho} \cdot u(\rho, \sigma)$ and

$$\sigma < u(\rho, \sigma) = \mathrm{cov}(\sigma, \sigma, \rho, 2) \cdot \sup_{\rho\le\nu<\sigma} u(\rho, \nu).$$

So we have

$$\mathrm{cov}(\sigma, \sigma, \rho, 2) = \sigma^{<\rho} > \sigma^+ = u(\sigma^+, \sigma^+). \quad ∎$$

In particular, if $2^{\aleph_0} < \aleph_\omega$ and $\aleph_{\omega+1} < \aleph_\omega^{\aleph_0}$, then $\mathcal{B}^{\omega_\omega}_{\omega_n,\omega_{\omega+1}}$ holds for all n with $0 < n < \omega$.

By work of Todorcevic [12] and of Cummings, Foreman and Magidor [1], if σ is a singular infinite cardinal, and $\square^*_\sigma$ holds (or there is a very good scale on σ), then one can find $y_\alpha \subseteq \sigma$ for $\alpha < \sigma^+$ so that (a) for every $\alpha < \sigma^+$, $\cup y_\alpha = \sigma$ and o.t. $(y_\alpha) = cf(\sigma)$, and (b) given $\beta < \sigma^+$, there is $g\colon \beta \longrightarrow \sigma$ such that

$$(y_\alpha \setminus g(\alpha)) \cap (y_{\alpha'} \setminus g(\alpha')) = \emptyset$$

for any $\alpha, \alpha' \in \beta$ with $\alpha \neq \alpha'$. As an immediate consequence we get:

Proposition 6.4: *Let σ be a cardinal such that $cf(\sigma) < \kappa < \sigma$ and $\square^*_\sigma$ holds. Then $\mathcal{B}^{\sigma,(cf(\sigma))^+}_{\kappa,\sigma^+}$ holds.*

The rest of the section is devoted to the proof of the result of Todorcevic [13] that $\omega_{\omega+1} \longrightarrow [\omega_1]^2_{\omega_\omega,<\omega_1}$ implies the failure of $\mathcal{A}^{\omega_\omega}_{\omega_1,\omega_{\omega+1}}$. For the consistency of $\omega_{\omega+1} \longrightarrow [\omega_1]^2_{\omega_\omega,<\omega_1}$ see [6].

Lemma 6.5: *Let τ be a cardinal such that $\kappa \leq \tau < \lambda$ and $\lambda \longrightarrow [\kappa]^2_{\tau,<\kappa}$, and let $C \subseteq P(\tau)$ with $|C| = \lambda$. Then there is $b \in P_\kappa(\tau)$ such that $|\{c \cap b : c \in C\}| \geq \kappa$.*

Proof: Select a bijection $j : \lambda \longrightarrow C$. Define $F\colon [\lambda]^2 \longrightarrow \tau$ so that $F(\{\alpha,\beta\}) \in j(\alpha)\Delta j(\beta)$. Pick $e \subseteq \lambda$ so that $|e| = \kappa$ and $|\{F(x) : x \in [e]^2\}| < \kappa$. Then $b = \{F(x) : x \in [e]^2\}$ is as desired. ∎

Lemma 6.6: *Let ν and σ be two cardinals such that $\omega \leq \nu < \kappa < \sigma < \lambda$ and $\mathcal{A}^{\sigma,\nu^+}_{\kappa,\lambda}$ holds. Then there is $C \subseteq \{c \subseteq \sigma^{<\nu} : |c| = cf(\nu)\}$ such that $|C| = \lambda$ and $|\{c \in C : |c \cap b| = cf(\nu)\}| < \kappa$ for every $b \in P_\kappa(\sigma^{<\nu})$ (and hence $\mathcal{A}^{\sigma^{<\nu},(cf(\nu))^+}_{\kappa,\lambda}$ holds).*

Proof: Since $\mathcal{A}^{\sigma,\nu^+}_{\kappa,\lambda}$ holds, there is $A \subseteq P_{\nu^+}(\sigma \setminus \kappa)$ such that $|A| = \lambda$ and $|\cup x| = \kappa$ for every $x \subseteq A$ with $|x| = \kappa$. Fix a strictly increasing sequence $< \eta_\xi : \xi < cf(\nu) >$ of ordinals with $\sup_{\xi<cf(\nu)} \eta_\xi = \nu$. For $a \in A$, select a bijection $j_a\colon \nu \longrightarrow a \cup \nu$ and put $\tilde{a} = \{j_a \restriction \nu_\xi : \xi < cf(\nu)\}$. Clearly, $B = \{\tilde{a} : a \in A\}$ has size λ. Now let $d \in P_\kappa(\bigcup_{\xi<cf(\nu)} \mathcal{F}_\xi)$, where $\mathcal{F}_\xi$ is the set of all functions from ν_ξ to σ. Set $z = \bigcup_{t\in d} \mathrm{ran}(t)$. Then $z \in P_\kappa(\sigma)$. Moreover, for each $a \in A$, $|\tilde{a} \cap d| = cf(\nu)$ implies that $a \subseteq z$. Hence $|\{b \in B : |b \cap d| = cf(\nu)\}| < \kappa$. The desired conclusion easily follows. ∎

PROPOSITION 6.7: *Let ν and σ be two cardinals such that* (a) $\omega \le \nu < \kappa < \sigma$, (b) $\sigma^{<\nu} < \lambda$, (c) $\mu^{<cf(\nu)} < \kappa$ *for every cardinal* $\mu < \kappa$ *and* (d) $\lambda \longrightarrow [\kappa]^2_{\sigma^{<\nu},<\kappa}$. *Then $\mathcal{A}^{\sigma,\nu^+}_{\kappa,\lambda}$ does not hold.*

Proof: By Lemmas 6.5 and 6.6.

COROLLARY 6.8: *Let σ be a cardinal such that $\omega_1 \le \sigma < \lambda$ and $\lambda \longrightarrow [\omega_1]^2_{\sigma,<\omega_1}$. Then $\mathcal{A}^{\sigma}_{\omega_1,\lambda}$ does not hold.*

7. $I_{\kappa,\lambda}|A$

In this section we deal with the question of whether for $\delta \ge \kappa$, there is A such that $NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$, or even $NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$. Our key tool for getting positive results is the following abstract version of a result of Baumgartner (Theorem 2.3 in [4]).

LEMMA 7.1: *Let I and J be two ideals on $P_\kappa(\lambda)$ such that $I \subseteq J$. Assume that for any $\mathcal{B} \subseteq J$ with $|\mathcal{B}| = \overline{\mathrm{cof}}(J)$, there is $D \in J^+$ such that $D \cap B \in I$ for every $B \in \mathcal{B}$. Then there is $A \in J^+$ such that $J|A = I|A$.*

Proof: Select $\mathcal{B} \subseteq J$ so that $|\mathcal{B}| = \overline{\mathrm{cof}}(J)$ and for every $C \in J$, there is $x \in P_\kappa(\mathcal{B})$ with $C \subseteq \cup x$. Now let $A \in J^+$ be such that $A \cap B \in I$ for all $B \in \mathcal{B}$. Given $C \in J \cap P(A)$, select $x \in P_\kappa(\mathcal{B})$ so that $C \subseteq \cup x$. Then $C \subseteq A \cap (\cup x)$, and since $A \cap (\cup x)$ belongs to I, so does C. Hence $I^+ \cap P(A) \subseteq J^+$. ∎

PROPOSITION 7.2: *Assume $\delta \ge \kappa$, and let J be an ideal on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) \le |\delta|^{<\overline{\theta}}$ and $P_\kappa(\lambda) \notin \nabla^{[\delta]^{<\theta}} J$. Then there is $A \in (\nabla^{[\delta]^{<\theta}} J)^*$ such that $J|A = I_{\kappa,\lambda}|A$.*

Proof: If $B_e \in J$ for $e \in P_{\overline{\theta}}(\delta)$, then $A \cap B_e \in I_{\kappa,\lambda}$ for all $e \in P_{\overline{\theta}}(\delta)$, where $A = P_\kappa(\lambda) - (\nabla_{d \in P_{\overline{\theta}}(\delta)} B_d)$. So the desired assertion can be inferred from Lemma 7.1. ∎

COROLLARY 7.3: *Let ζ be an ordinal with $\kappa \le \zeta \le \delta$, and η be a cardinal with $2 \le \eta \le \theta$. Assume that there exists $C \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^*$ such that $\overline{\mathrm{cof}}(NS^{[\zeta]^{<\eta}}_{\kappa,\lambda}|C) \le |\delta|^{<\overline{\theta}}$. Then $NS^{[\zeta]^{<\eta}}_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some $A \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^*$.*

Proof: Set $J = NS^{[\zeta]^{<\eta}}_{\kappa,\lambda}|C$. We have

$$\nabla^{[\delta]^{<\theta}} J \subseteq \nabla^{[\delta]^{<\theta}} (NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|C) = \nabla^{[\delta]^{<\theta}} NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = NS^{[\delta]^{<\theta}}_{\kappa,\lambda}.$$

Hence, by Proposition 7.2, there is $D \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^*$ such that $J|D = I_{\kappa,\lambda}|D$. Now setting $A = C \cap D$, we get

$$NS_{\kappa,\lambda}^{[\zeta]^{<\eta}}|A = (J|D)|C = (I_{\kappa,\lambda}|D)|C = I_{\kappa,\lambda}|A. \quad \blacksquare$$

Corollary 7.4: *Assume that $\delta \geq \kappa$ and $\lambda^{(|\delta|^{<\overline{\theta}})} = \lambda^{<\overline{\theta}}$. Then $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A = I_{\kappa,\lambda}|A$ for some $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$.*

Proof: The result follows immediately from Corollary 7.3 since $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \leq \mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \leq \lambda^{(|\delta|^{<\overline{\theta}})}$. ∎

Corollary 7.5: *Assume that $\delta \geq \kappa$ and J is a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(J) \leq |\delta|^{<\overline{\theta}}$. Then $J = I_{\kappa,\lambda}|A$ for some $A \in I_{\kappa,\lambda}^+$.*

Proof: This is an immediate consequence of Proposition 7.2. ∎

Suppose $\delta \geq \kappa$. If there is $A \in I_{\kappa,\lambda}^+$ such that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}|A$, then by a result of [7], $|\delta|^{<\overline{\theta}} \geq \lambda$. From this and Corollary 7.5 we can conclude that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \leq |\delta|^{<\overline{\theta}}$ if and only if $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}|A$ for some A.

Proposition 7.6: *Assume that λ is a strong limit cardinal and $\overline{\theta} \leq cf(\lambda) < \kappa$. Then $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} = I_{\kappa,\lambda}|A$ for some $A \in I_{\kappa,\lambda}^+$.*

Proof: By Proposition 3.6 and Corollary 7.5. ∎

Corollary 7.5 can also be used to obtain a lower bound for $\overline{\mathrm{cof}}(NS_{\kappa,\lambda})$.

Proposition 7.7:

(i) *Let σ be the least cardinal τ such that $\tau^{<\overline{\theta}} \geq \lambda$. Assume $\delta \geq \sigma$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \geq \sigma$.*

(ii) *Assume that $cf(\lambda) \geq \kappa$ and $\mu^{<\overline{\theta}} < \lambda$ for every cardinal $\mu < \lambda$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}) > \lambda^{<\overline{\theta}}$.*

Proof:

(i) Suppose otherwise. Then by Corollary 7.5, there exists $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^*$ such that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}|A$. Now Proposition 5.2 (iii) tells us that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \geq \sigma$, which is a contradiction.

(ii) Supppose otherwise. Then by Corollary 7.5, there is $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$ such that $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} = I_{\kappa,\lambda}|A$. Now Proposition 5.2 (iii) says that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) = \lambda$, contradicting Proposition 4.3. ∎

In particular, $\overline{\text{cof}}(NS_{\kappa,\lambda}) \geq \lambda$. Moreover, this inequality is strict in case $cf(\lambda) \geq \kappa$.

If $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A = I_{\kappa,\lambda}|A$, then clearly $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) \leq \lambda$. Let us next discuss the problem whether for $\delta \geq \kappa$, there is A such that $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) < \lambda$.

LEMMA 7.8: *Let σ be a cardinal with $\kappa \leq \sigma < \lambda$, ζ be an ordinal with $\delta \leq \zeta \leq \lambda$, and η be a cardinal with $\theta \leq \eta \leq \kappa$. Assume that* (a) $\delta \geq \kappa$, (b) *there is $D \in (NS_{\kappa,\lambda}^{[\zeta]^{<\eta}})^*$ such that $\overline{\text{cof}}(I_{\kappa,\lambda}|D) \leq \sigma$, and* (c) *there is $C \in (NS_{\kappa,\lambda}^{[\zeta]^{<\eta}})^*$ such that $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|C) \leq |\zeta|^{<\overline{\eta}}$. Then there is $B \in (NS_{\kappa,\lambda}^{[\zeta]^{<\eta}})^*$ such that $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) \leq \sigma$.*

Proof: We can apply Corollary 7.3 and obtain $A \in (NS_{\kappa,\lambda}^{[\zeta]^{<\eta}})^*$ such that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A = I_{\kappa,\lambda}|A$. Setting $B = D \cap A$, we get $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) \leq \overline{\text{cof}}(I_{\kappa,\lambda}|D) \leq \sigma$. ∎

LEMMA 7.9: *Assume that $\delta \geq \kappa$, $\lambda = \sigma^+$ and $|\delta|^{<\overline{\theta}} \leq \sigma$. Then there is $A \in NS_{\kappa,\lambda}^*$ such that for every $B \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+ \cap P(A)$, $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) \leq \overline{\text{cof}}(I_{\kappa,\lambda}|B) \cdot \overline{\text{cof}}(NS_{\kappa,\sigma}^{[\delta\cap\sigma]^{<\theta}})$.*

Proof: For $\gamma < \lambda$, select two bijections $j_\gamma\colon \gamma \cap \sigma \longrightarrow \gamma$ and $k_\gamma\colon P_2(\sigma) \longrightarrow P_2(\gamma \cup \sigma)$ so that

(i) If $\gamma \leq \sigma$, then j_γ is the identity on γ, and k_γ the identity on $P_2(\sigma)$.
(ii) If $\gamma > \sigma$, then $k_\gamma(\emptyset) = \emptyset$ and $k_\gamma(\{\zeta\}) = \{j_\gamma(\zeta)\}$.

Let q denote the inverse of k_δ. Set $W = \{a \in P_\kappa(\lambda) : a \cap \kappa \in \kappa\}$ and

$$A = \widehat{\{\delta\}} \cap W \cap C_q^{\kappa,\lambda} \cap (\Delta_{\gamma\in\lambda} C_{k_\gamma}^{\kappa,\lambda}).$$

We have $W \in (NS_{\kappa,\lambda}^{\kappa})^*$, $C_q^{\kappa,\lambda} \in (NS_{\kappa,\lambda}^{\delta\cup\sigma})^*$ and for every $\gamma \in \lambda$, $C_{k_\gamma}^{\kappa,\lambda} \in (NS_{\kappa,\lambda}^{\sigma})^*$. Hence A belongs to $NS_{\kappa,\lambda}^*$ (and so to $(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$). Select a collection $\mathcal{F}$ of functions from $P_{\overline{\theta}\cdot 3}(\delta \cap \sigma)$ to $P_3(\sigma)$ so that $|\mathcal{F}| = \overline{\text{cof}}(NS_{\kappa,\sigma}^{[\delta\cap\sigma]^{<\theta}})$ and for every $g\colon P_{\overline{\theta}\cdot 3}(\delta \cap \sigma) \longrightarrow P_3(\sigma)$, there is $x \in P_\kappa(\mathcal{F}) \setminus \{\emptyset\}$ with

$$\bigcap_{f\in x} \{a \in C_f^{\kappa,\sigma} : a \cap \kappa \in \kappa\} \subseteq C_g^{\kappa,\lambda}.$$

For $f \in \mathcal{F}$, define $\overline{f}\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_3(\sigma)$ by $\overline{f}(e) = f(j_\delta^{-1}(e))$.

Now let $h\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_3(\lambda)$. Pick $\gamma \in \lambda$ so that $h(e) \subseteq \gamma$ for all $e \in P_{\overline{\theta}\cdot 3}(\delta)$. Define $g\colon P_{\overline{\theta}\cdot 3}(\delta \cap \sigma) \longrightarrow P_3(\gamma \cap \sigma)$ by $g(d) = j_\gamma^{-1}(h(j_\delta[d]))$. Select

$x \in P_\kappa(\mathcal{F}) \setminus \{\emptyset\}$ so that $\bigcap_{f\in x}\{a \in C_f^{\kappa,\sigma} : a \cap \kappa \in \kappa\} \subseteq C_g^{\kappa,\lambda}$. Set $Y = A \cap \widehat{\{\gamma\}} \cap \bigcap_{f\in x} C_{\overline{f}}^{\kappa,\lambda}$. We claim that $Y \subseteq C_h^{\kappa,\lambda}$. To prove the claim, let $b \in Y$ and set $a = b \cap \sigma$. Obviously, $a \cap (\overline{\theta} \cdot 3) = b \cap (\overline{\theta} \cdot 3)$ and $a \cap \kappa \in \kappa$. Let $f \in x$ and $d \in P_{|a\cap(\overline{\theta}\cdot 3)|}(a \cap (\delta \cap \sigma))$. We have $j_\delta[d] \in P_{|b\cap(\overline{\theta}\cdot 3)|}(b \cap \delta)$, since $b \in C_{k_\delta}^{\kappa,\lambda}$. So it follows from $b \in C_{\overline{f}}^{\kappa,\lambda}$ that $f(d) \subseteq b \cap \sigma$. Thus $a \in \bigcap_{f\in x} C_f^{\kappa,\sigma}$, hence $a \in C_g^{\kappa,\sigma}$. Now let $e \in P_{|b\cap(\overline{\theta}\cdot 3)|}(b \cap \delta)$. Since $b \in C_q^{\kappa,\lambda}$, we have $j_\delta^{-1}(e) \in P_{|a\cap(\overline{\theta}\cdot 3)|}(a\cap(\delta\cap\sigma))$. From $a \in C_g^{\kappa,\sigma}$, we can infer that $j_\gamma^{-1}(h(e)) \subseteq a$. It follows that $h(e) \subseteq b$, since $b \in C_{k_\gamma}^{\kappa,\lambda}$. This completes the proof of the claim. Now given $B \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+ \cap P(A)$, we get

$$B \cap \widehat{\{\gamma\}} \cap \bigcap_{f\in x} C_{\overline{f}}^{\kappa,\lambda} \subseteq B \cap C_h^{\kappa,\lambda}.$$

Consequently, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) \leq |\mathcal{F}| \cdot \overline{\mathrm{cof}}(I_{\kappa,\lambda}|B)$. ∎

Proposition 7.10: *Let σ be a strong limit cardinal, and let $\tau = (cf(\sigma))^+$. Assume that $\theta < \tau < \kappa < \sigma \leq \delta < \sigma^+ \leq \lambda \leq 2^\sigma$ and there exists a $[\sigma]^{<\tau}$- normal ideal on $P_\kappa(\lambda)$. Then there is $T \in (NS_{\kappa,\lambda}^{[\delta]^{<\tau}})^*$ such that* (a) $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|T = I_{\kappa,\lambda}|T$, *and* (b) $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|T) = \sigma$.

Proof: By Proposition 5.3, there is $D \in (NS_{\kappa,\lambda}^{[\sigma]^{<\tau}})^*$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) = \sigma$. Furthermore, Propositions 3.1 and 4.1 (iv) yield $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \leq \sigma \cdot \lambda^\sigma = |\delta|^{<\tau}$. Therefore, by Lemma 7.8, there is $B \in (NS_{\kappa,\lambda}^{[\delta]^{<\tau}})^*$ such that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) = \sigma$. From Corollary 7.5 we can infer that there is $C \in I_{\kappa,\lambda}^+$ such that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B = I_{\kappa,\lambda}|C$. Then $B\setminus C \in NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$, hence $P_\kappa(\lambda)\setminus C \in NS_{\kappa,\lambda}^{[\delta]^{<\tau}}$. So setting $T = B\cap C$, we have $T \in (NS_{\kappa,\lambda}^{[\delta]^{<\tau}})^*$ and $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|T = I_{\kappa,\lambda}|T$. Proposition 5.2 (iii) gives $\sigma \leq \overline{\mathrm{cof}}(I_{\kappa,\lambda}|T)$. Conversely, $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|T) \leq \sigma$ is true because $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|T) \leq \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) = \sigma$. ∎

Note that if σ is a strong limit cardinal such that $\overline{\theta} \leq cf(\sigma) < \kappa < \sigma \leq \delta < \sigma^+ \leq \lambda < \sigma^{+\kappa}$, then by Corollary 3.2, Corollary 3.3 and Proposition 4.1 (iv), we have $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \lambda$, hence by Corollary 7.3 and Proposition 5.2 (iii), there is $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$ such that (a) $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A = I_{\kappa,\lambda}|A$, and (b) $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) = \lambda$.

Proposition 7.11: *Let σ be a strong limit cardinal, and let $\tau = (cf(\sigma))^+$. Assume that $\tau < \kappa \leq \delta < \sigma$ and there exists a $[\sigma]^{<\tau}$-normal ideal on $P_\kappa(\lambda)$. Then*

(i) *If* $\sigma < \lambda \le 2^\sigma$, *then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) = \sigma$ *for some* $B \in (NS_{\kappa,\lambda}^{[\sigma]^{<\theta\cdot\tau}})^*$.

(ii) *If* $\lambda = \sigma^+$ *and* $\tau < \theta$, *then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) = \sigma$ *for some* $B \in (NS_{\kappa,\lambda}^{[\lambda]^{<\tau}})^*$.

Proof:

(i) Suppose $\sigma < \lambda \le 2^\sigma$. Then by Proposition 5.3, there is D in $(NS_{\kappa,\lambda}^{[\sigma]^{<\tau}})^*$ (and hence in $(NS_{\kappa,\lambda}^{[\sigma]^{<\theta\cdot\tau}})^*$) such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) = \sigma$. Furthermore, we have

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \le \mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) \le \lambda^{(|\delta|^{<\overline{\theta}})} \le \lambda^\sigma = \sigma^{<\theta\cdot\tau}.$$

Now the assertion follows from Lemma 7.8.

(ii) Suppose $\lambda = \sigma^+$ and $\tau < \theta$. Then by Lemma 7.9 and Proposition 4.1 (iii), there is $A \in NS_{\kappa,\lambda}^*$ such that for every $B \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+ \cap P(A)$, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) \le \sigma \cdot \overline{\mathrm{cof}}(I_{\kappa,\lambda}|B)$. Moreover, by Proposition 5.3, there is $D \in (NS_{\kappa,\lambda}^{[\sigma]^{<\tau}})^*$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) = \sigma$. Now put $B = A \cap D$. Obviously, $B \in (NS_{\kappa,\lambda}^{[\lambda]^{<\tau}})^*$. Proposition 4.7 gives $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) \ge \sigma$. On the other hand, we have $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B) \le \sigma \cdot \overline{\mathrm{cof}}(I_{\kappa,\lambda}|D) = \sigma$. ∎

With GCH, we obtain the following picture.

PROPOSITION 7.12: *Assume that the* GCH *holds and* $\delta \ge \kappa$. *Then*

(i) *If* $\delta = \lambda$ *and* $cf(\lambda) < \overline{\theta}$, *then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda^{++}$ *for all* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.

(ii) *If* $\kappa \le cf(\lambda) \le |\delta|^{<\overline{\theta}}$, *then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda^+$ *for all* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.

(iii) *Assume that* (a) $\delta < \lambda$ *and* $cf(\lambda) < \kappa$, *or* (b) $\delta = \lambda$ *and* $\overline{\theta} \le cf(\lambda) < \kappa$, *or* (c) $\delta < \lambda = \sigma^+$ *and* $cf(\sigma) \ge \kappa$, *or* (d) λ *is a limit cardinal and* $|\delta|^{<\overline{\theta}} < cf(\lambda)$. *Then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda$ *for all* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.

(iv) *Assume that* $\lambda = \sigma^+$ *and either* $\delta < \sigma$ *and* $cf(\sigma) < \overline{\theta}$, *or* $\delta < \lambda$, $\overline{\theta} \le cf(\sigma) < \kappa$ *and* κ *is not the successor of a cardinal of cofinality less than or equal to* $cf(\sigma)$. *Then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \sigma$ *for some* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.

Proof:

(i) Suppose $cf(\lambda) < \overline{\theta}$, and let $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^+$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) \le \mathrm{cof}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) = \lambda^{++}$. Furthermore, we have $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}|A) > \lambda^+$ since $u(\kappa, \lambda^+) < \lambda^{++}$.

(ii) Assume $\kappa \le cf(\lambda) \le |\delta|^{<\overline{\theta}}$, and fix $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) \le \mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda^+$. But clearly we have $u(\kappa, \lambda) < \lambda^+$, which gives $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) > \lambda$.

(iii) By Proposition 4.1.

(iv) By Propositions 7.10 and 7.11. ∎

Let us next consider the case that was not dealt with in Proposition 7.12, namely the case when $\lambda = \sigma^+$, $\kappa = \nu^+$, $\overline{\theta} \cdot cf(\nu) \leq cf(\sigma) < \kappa$ and $\delta < \lambda$.

PROPOSITION 7.13: *Let θ' be a cardinal with $\theta \leq \theta' \leq \kappa$. Assume that (a) $\lambda = \sigma^+$, (b) $\sigma > \kappa$, (c) either $\mathcal{A}^{\sigma,\rho}_{\kappa,\lambda}$ holds for some regular infinite cardinal $\rho < \kappa$, or $\mathcal{B}^{\sigma}_{\kappa,\lambda}$ holds, (d) either $\kappa \leq \delta < \sigma$ and $\tau^{(|\delta|^{<\overline{\theta}})} < \sigma$ for every cardinal $\tau < \sigma$, or $\sigma \leq \delta < \lambda$, $\overline{\theta} \leq cf(\sigma) < \kappa$ and σ is a strong limit cardinal, and (e) there exists a $[\lambda]^{<\theta'}$-normal ideal on $P_\kappa(\lambda)$. Then there is $B \in (NS^{[\lambda]^{<\theta'}}_{\kappa,\lambda})^+$ such that $\overline{\text{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|B) = \sigma$.*

Proof: By Propositions 5.4 (ii) and 5.11, there is $D \in (NS^{[\lambda]^{<\theta'}}_{\kappa,\lambda})^+$ such that $\overline{\text{cof}}(I_{\kappa,\lambda}|D) \leq \sigma$. From Lemma 7.9 we obtain $A \in NS^*_{\kappa,\lambda}$ such that for every $B \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+ \cap P(A)$,

$$\overline{\text{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|B) \leq \overline{\text{cof}}(I_{\kappa,\lambda}|B) \cdot \overline{\text{cof}}(NS^{[\delta\cap\sigma]^{<\theta}}_{\kappa,\sigma}).$$

Now put $B = A \cap D$. By Proposition 4.1 ((iii) and (iv)) and Corollary 3.2, we have $\overline{\text{cof}}(NS^{[\delta\cap\sigma]^{<\theta}}_{\kappa,\sigma}) = \sigma$. Therefore, $\overline{\text{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|B) \leq \overline{\text{cof}}(I_{\kappa,\lambda}|D) \cdot \sigma = \sigma$. Also, $\overline{\text{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|B) \geq \sigma$, since $u(\kappa,\nu) < \lambda$ for every cardinal ν with $\kappa \leq \nu < \sigma$. ∎

In particular, with the help of Proposition 6.4, we have: Assume that (a) $\kappa \leq \delta < \lambda = \sigma^+$, (b) σ is a strong limit cardinal with $cf(\sigma) < \kappa$, and (c) $\square^*_\sigma$ holds. Then $\overline{\text{cof}}(NS^{\delta}_{\kappa,\lambda}|B) = \sigma$ for some $B \in NS^+_{\kappa,\lambda}$. On the other hand, by Proposition 5.7 and Corollary 6.8, $\omega_{\omega+1} \longrightarrow [\omega_1]^2_{\omega_\omega,<\omega_1}$ implies that $\overline{\text{cof}}(NS^{\delta}_{\omega_1,\omega_{\omega+1}}|B) > \omega_\omega$ for each δ with $\omega_1 \leq \delta < \omega_{\omega+1}$ and each $B \in (NS^{\delta}_{\omega_1,\omega_{\omega+1}})^+$.

If the GCH is assumed, then our question of the existence of sets B such that $NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|B = I_{\kappa,\lambda}|B$ can be answered completely.

PROPOSITION 7.14: *Assume that the GCH holds and $\delta \geq \kappa$. Let χ_0 and χ_1 denote, respectively, the assertions "$NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$ for some $A \in I^+_{\kappa,\lambda}$" and*

"$NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B = I_{\kappa,\lambda}|B$ *for some* $B \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$". *Then*

	$\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$	$\text{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$	χ_0	χ_1
$\lvert\delta\rvert^{<\overline{\theta}} < cf(\lambda)$	λ	λ	*no*	*yes*
$\kappa \leq cf(\lambda) \leq \lvert\delta\rvert^{<\overline{\theta}}$	λ^+	λ^+		*no*
$\delta < \lambda$ *and* $cf(\lambda) < \kappa$	λ	λ^+	*no*	*yes*
$\delta = \lambda$ *and* $cf(\lambda) < \overline{\theta}$	λ^{++}	λ^{++}		*no*
$\delta = \lambda$ *and* $\overline{\theta} \leq cf(\lambda) < \kappa$	λ	λ^+	*yes*	

Proof: See Propositions 4.5, 7.12 ((ii), (iii) and (i)) and 4.1 (iv) for the value of $\overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$, and [7] for that of $\text{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$. If $|\delta|^{<\overline{\theta}} < cf(\lambda)$, or $\delta < \lambda$ and $cf(\lambda) < \kappa$, then by Corollary 7.3, χ_1 holds, but by a result of [7], χ_0 fails. If $\delta = \lambda$ and $\overline{\theta} \leq cf(\lambda) < \kappa$, then χ_0 holds by Proposition 7.6. Finally, if either $\kappa \leq cf(\lambda) \leq |\delta|^{<\overline{\theta}}$, or $\delta = \lambda$ and $cf(\lambda) < \overline{\theta}$, then $\overline{\text{cof}}(I_{\kappa,\lambda}|B) < \overline{\text{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|B)$ for every $B \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$, hence χ_1 does not hold. ∎

8. Cohen forcing

In this final section we construct a forcing extension in which $NS_{\kappa,\lambda}^{[\zeta]^{<\theta}}|A \neq I_{\kappa,\lambda}|A$ for each ζ with $\kappa \leq \zeta \leq \lambda$, and each $A \in (NS_{\kappa,\lambda}^{[\zeta]^{<\theta}})^+$.

LEMMA 8.1: *Let R be a κ-closed notion of forcing. If there exists (in V) a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, then the same holds in V^R.*

Proof: This is clear from Proposition 1.5 (i) in case $\theta < \kappa$ or κ is not a limit cardinal. Otherwise, use Proposition 1.5 (ii) and the fact (see, e.g., Exercise H4 in Chapter VII of [5]) that if κ is Mahlo in V, then κ remains Mahlo in V^R. ∎

LEMMA 8.2: *Let μ be a cardinal such that $\kappa \cdot (|\delta|^{<\overline{\theta}})^+ \leq \mu = \mu^{<\mu} \leq \lambda$, and Q be the notion of forcing which adds a Cohen subset of μ. Further, let $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$. Then in V^Q, $P(A) \cap (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+ \cap \nabla^\mu I_{\kappa,\lambda} \neq \emptyset$.*

Proof: Q can be taken to be the set of all functions q such that $\text{dom}(q) \in P_\mu(\mu \times \mu)$ and $\text{ran}(q) \subseteq 2$. For a Q-generic set G over V, define $F_G\colon \mu \longrightarrow P_2(\mu)$

as follows. Given $\alpha \in \mu$, put $e_\alpha = \{\beta \in \mu : (\cup G)(\alpha,\beta) = 1\}$. Now set $F_G(\alpha) = \{\cap e_\alpha\}$ if $e_\alpha \neq \emptyset$, and $F_G(\alpha) = \emptyset$ otherwise.

Let us show that $\Vdash_Q A \setminus C^{\kappa,\lambda}_{F_{\dot G}} \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+$. Thus, let $q \in Q$ and $f\colon P_{\overline{\theta}\cdot 3}(\delta) \longrightarrow P_\kappa(\lambda)$. Pick $\alpha \in \mu$, $a \in A$ and $\beta \in \mu$ so that $(\{\alpha\} \times \mu) \cap \mathrm{dom}(q) = \emptyset$, $\alpha \in a \in A \cap C^{\kappa,\lambda}_f$ and $\beta \notin a$. Now select $r \in Q$ so that $q \subseteq r$, $r(\alpha,\beta) = 1$, and $r(\alpha,\gamma) = 0$ for all $\gamma < \beta$. Then clearly, $r \Vdash b \notin C^{\kappa,\lambda}_{F_{\dot G}}$. ∎

Suppose that μ is a cardinal such that $\kappa \leq \mu = \mu^{<\mu} \leq \lambda$, and for every $Z \subseteq \lambda^{<\kappa}$ with $P_\mu(\mu) \subseteq L[Z]$, there is a subset of μ which is Cohen over $L[Z]$. Then $NS^\mu_{\kappa,\lambda}|A \neq I_{\kappa,\lambda}|A$ for every $A \in (NS^\mu_{\kappa,\lambda})^+$. To see this, fix $A \in (NS^\mu_{\kappa,\lambda})^+$. Select $Z \subseteq \lambda^{<\kappa}$ so that $A \in L[Z]$, $P_\mu(\mu) \subseteq L[Z]$ and $P_\kappa(\lambda) \subseteq L[Z]$. Let $G \subseteq \mu$ be Cohen over $L[Z]$. By Lemma 8.1, in $L[Z][G]$ we can find $C \in I^+_{\kappa,\lambda} \cap P(A)$ and $g\colon P_3(\mu) \longrightarrow P_3(\lambda)$ so that $C \cap \{a \in C^{\kappa,\lambda}_g : a \cap \kappa \in \kappa\} = \emptyset$. Now C and g are like this in V, so we are done.

Proposition 8.3: *Let μ be a cardinal such that $\kappa \cdot (|\delta|^{<\overline{\theta}})^+ \leq \mu = \mu^{<\mu} \leq \lambda$, ρ be a cardinal such that $\lambda^{<\kappa} < \rho$, and P be the notion of forcing which adds ρ Cohen subsets of μ. Then in V^P, $NS^{[\mu]^{<\theta}}_{\kappa,\lambda}|A \neq NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A$ for all $A \in (NS^{[\mu]^{<\theta}}_{\kappa,\lambda})^+$.*

Proof: P can be identified with the set of all functions p such that $\mathrm{dom}(p) \in P_\mu(\rho \times \mu)$ and $\mathrm{ran}(p) \subseteq 2$. Now let G be P-generic over V. For $X \subseteq \rho$, set $G_X = \{p \in G : \mathrm{dom}(p) \subseteq X \times \mu\}$. In $V[G]$, let $A \in (NS^{[\mu]^{<\theta}}_{\kappa,\lambda})^+$. Then there is $\xi < \rho$ with $A \in V[G_\xi]$. From Lemma 8.2 we can infer that in $V[G_\xi][G_{\{\xi\}}]$, $P(A) \cap (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+ \cap NS^{[\mu]^{<\theta}}_{\kappa,\lambda} \neq \emptyset$. The same inequality must hold in $V[G]$. ∎

Corollary 8.4: *Assume that $2^{<\kappa} = \kappa \leq \delta$. Let ρ be a cardinal such that $\lambda^{<\kappa} < \rho$, and P be the notion of forcing which adds ρ Cohen subsets of κ. Then in V^P,* (a) *$NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A \neq I_{\kappa,\lambda}|A$ for all $A \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+$,* (b) *$\overline{\mathrm{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|B) > \lambda^{<\overline{\theta}}$ for all $B \in (NS^{[\lambda]^{<\theta}}_{\kappa,\lambda})^+$.*

Proof: By Proposition 8.3 we have that in V^P, $NS^\kappa_{\kappa,\lambda}|A \neq I_{\kappa,\lambda}|A$ for all $A \in (NS^\kappa_{\kappa,\lambda})^+$. Part (a) follows, since $NS^\kappa_{\kappa,\lambda} \subseteq NS^{[\delta]^{<\theta}}_{\kappa,\lambda}$. For (b) use Proposition 7.2. ∎

References

[1] J. Cummings, M. Foreman and M. Magidor, *Squares, scales and stationary reflection,* Journal of Mathematical Logic **1** (2001), 35–98.

[2] H. D. Donder, P. Koepke and J. P. Levinski, Personal communication, June 1998.

[3] H. D. Donder and P. Matet, *Two cardinal versions of diamond,* Israel Journal of Mathematics **83** (1993), 1–43.

[4] C. A. Johnson, *Seminormal λ-generated ideals on $P_\kappa\lambda$,* Journal of Symbolic Logic **53** (1988), 92–102.

[5] K. Kunen, *Set Theory,* North-Holland, Amsterdam, 1980.

[6] J. P. Levinski, M. Magidor and S. Shelah, *Chang's Conjecture for $\aleph_\omega$,* Israel Journal of Mathematics **69** (1990), 161–172.

[7] P. Matet, C. Péan and S. Shelah, *Cofinality of normal ideals on $P_\kappa(\lambda)$ I,* preprint.

[8] S. Shelah, *Cardinal Arithmetic,* Oxford Logic Guides Vol. 29, Oxford University Press, Oxford, 1994.

[9] S. Shelah, *The generalized continuum hypothesis revisited,* Israel Journal of Mathematics **116** (2000), 285–321.

[10] S. Shelah, *On the existence of large subsets of $[\lambda]^{<\kappa}$ which contain no unbounded non-stationary subsets,* Archive for Mathematical Logic **41** (2002), 207–213.

[11] S. Shioya, *Generating the club filter on $P_\kappa\lambda$,* Topology and its Applications **122** (2002), 415–419.

[12] S. Todorcevic, *Coherent sequences,* in *Handbook of Set Theory* (M. Foreman, A. Kanamori and M. Magidor, eds.), Kluwer, Dordrecht, to appear.

[13] S. Todorcevic, Personal communication.